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Functional Inequalities for Bessel and Hypergeometric Type Functions via Probabilistic Approach

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To the memory of my uncle Rev. Pater J. Pogány (Gombos, Habsburg Empire 1909–Galle, Sri Lanka 1986)

Abstract. In this exposition article the investigation methodology is the following: **(i)** consider a probability density function (pdf) of a continuous random variable (rv), **(ii)** derive the related cumulative distribution function (cdf) and **(iii)** apply to the obtained expressions general suitable properties of cumulative distribution functions to get either functional or uniform bounds upon the special functions which are the building blocks of the cdf. However, this should agree with the title of the presentation, therefore we focus on a rv which cdf and/or pdf contain Bessel or some type of hypergeometric function.

The familiar inter-connection formula between the modified Bessel function of the first kind and the confluent hypergeometric function (Kummer function in other words) readily shows that the modified Bessel function can be transformed into a hypergeometric function of a special kind. Moreover, since the pdf is non-negative on the real line and normalized and the modified Bessel functions of the first kind $I_\nu(x)$ is non-negative for positive values of the argument, the McKay Bessel I_ν distribution exactly fits the requirements of the investigation.

Keywords. Confluent Horn Φ_2, Φ_3 functions, Exton X function, Functional bounding inequality, Grünwald–Letnikov fractional derivative, Incomplete Lipschitz-Hankel integral, Kampé de Fériet function, Lower incomplete Fox–Wright functions; McKay I_ν Bessel distribution, Modified Bessel functions of the first kind, Srivastava–Daoust $\mathcal{S}$ function, Turán inequality.

Subject Classification 2020: 26D15; 26D20; 33C10; 33C20; 33C70; 33E20; 44A60; 60E05.

1. Introduction

The probability distributions involving Bessel functions attracted attention of the statistical and/or mathematical community mainly after 1932 since the early work of McKay [28] in 1932 and Laha [24] in 1954 who considered two classes of continuous distributions called *Bessel function distributions*, see also [51] (and the references therein).

The probabilistic approach we begin with considering the random variable (rv) ξ defined on a standard probability space $(\Omega, \mathscr{A}, \mathsf{P})$ which has McKay Bessel law when the associated probability density function (pdf) reads

$$f_{I(K)}(x; b, c; \nu) = y_0 \mathrm{e}^{-\frac{cx}{b}} \, |x|^\nu \begin{cases} \pi I_\nu\left(\left|\dfrac{x}{b}\right|\right) \\ \\ K_\nu\left(\left|\dfrac{x}{b}\right|\right); \end{cases} \qquad x \in \mathbb{R}, \qquad (1.1)$$

where $b > 0$, $\nu > -1/2$ and

$$y_0 = \frac{|1 - c^2|^{\nu+\frac{1}{2}}}{\sqrt{\pi}\, 2^\nu b^{\nu+1} \Gamma\left(\nu + \frac{1}{2}\right)}.$$

The symbol I_ν denotes, throughout, the modified Bessel function of the first kind of order ν, which has the power series representation (see e.g. [20, p. 902, Eq. (1.2)], [34, p. 249, Eq. **10.25.2**])

$$I_\nu(x) = \sum_{n\geq 0} \frac{1}{\Gamma(\nu + n + 1)\, n!} \left(\frac{x}{2}\right)^{2n+\nu},$$

where $\Re(\nu) > -1$, $x \in \mathbb{C}$, whilst K_ν stands for the modified Bessel function of the second kind of order ν (called also Basset function or Macdonald function for which there is no such series representation formula in the general case). The upper function in (1.1), that is, the pdf of the rv distributed according to the McKay I_ν law is employed when $|c| > 1$, in which case the distribution curve extends from 0 to ∞ if c is positive, and from $-\infty$ to 0 when c is negative. Sometimes, the previous pdf (1.1) is called I-form curve.

In 1973 McNolty presented [29, p. 496, Eq. (13)], without derivation, several generalized pdfs together with the related characteristic functions, among others the pdf

$$f_I(x;a,b;\nu) = C_\nu\, x^\nu\, \mathrm{e}^{-bx}\, I_\nu(ax),$$

defined for all $\nu > -1/2$ and $b > a > 0$, $\mathrm{supp}(f_I) = \mathbb{R}_+$ and

$$C_\nu := C_\nu(a,b) = \frac{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}}{(2a)^\nu \Gamma\left(\nu+\frac12\right)}.$$

Having in mind the asymptotic of $I_\nu(x)$ when the argument approaches zero or infinity, the tail behaviour of the pdf reads, respectively [34, §10.25, §10.40]

$$f_I(x;a,b;\nu) \sim \begin{cases} \dfrac{(b^2-a^2)^{\nu+1/2}\, x^{2\nu}}{\Gamma(2\nu+1)}\left(1+\mathcal{O}(x^2)\right), \\ \dfrac{(b^2-a^2)^{\nu+1/2}\, x^{\nu-\frac12}}{(2a)^{\nu+\frac12}\,\Gamma(\nu+\frac12)\mathrm{e}^{(b-a)x}}\left(1+\mathcal{O}(x^{-1})\right). \end{cases}$$

The related cdf is of the form

$$F_I(x;a,b;\nu) = C_\nu \int_0^x t^\nu \mathrm{e}^{-bt} I_\nu(at)\,\mathrm{d}t, \quad x \ge 0. \qquad (1.2)$$

The properties of previously re-called distributions including its characteristic functions, estimation issues, the appropriate moments and moment generating functions and extensions have been subjects of several articles, for instance [29,32,33]. There are also results concerning applications of such functions, consult for instance, Bose [7] who used I-form curve for graduating a particular frequency distribution, similarly Bhattacharyya [4] applied I-form curves for related goals. In the electrical and electronic engineering literature that requires the study of further generalizations of previously listed pdfs, see e.g. [32].

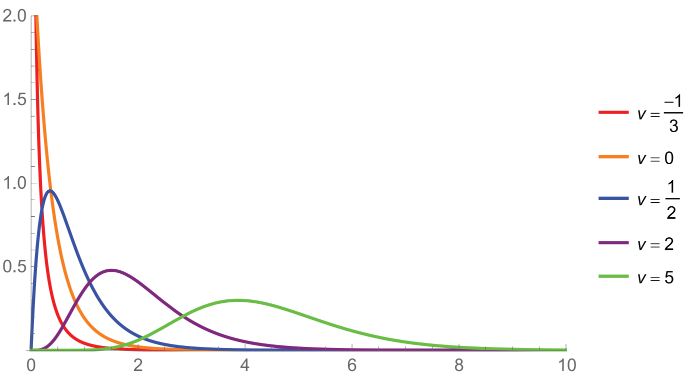

Figure 1. pdf $f_I(x;a,b;\nu)$ for $a = 1.2$, $b = 3$ and $\nu = -\frac13, 0, \frac12, 2, 5$. For $\nu < 0$ the pdfs tend to infinity as $x \to 0^+$; for $\nu > 0$ the pdfs vanish at $x = 0$, whilst $f_I(x;a,b;0) = \sqrt{b^2-a^2}$.

The fact that the rv ξ is distributed according to McKay I_ν Bessel law we write throughout as

$$\xi \sim \mathrm{McKayI}(a,b,\nu); \quad \xi \sim f_I; \quad \text{or} \quad \xi \sim F_I.$$

Our main task is now to establish closed form expressions for the cdf (1.2) and using certain properties of the cdf to obtain bounding inequalities for the special functions which build these cdfs. We point out that the main results of publications [14,19,20,21,22,37] coauthored with D. Jankov Maširević, K. Górska and A. Horzela consist the main parts of this expository article. At the end of the work in the discussion section a short survey is given for another kind inequalities for Bessel, modified Bessel and hypergeometric type functions, whose derivation is *not* based on probabilistic methods.

2. Hypergeometric type functions review + ILHI, GLFD and cdf inequalities

2.1 Gamma function and Pochhammer symbol

One of the main auxiliary tools is the (generalized) Pochhammer symbol

$$(\lambda)_\mu = \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)} = \begin{cases} 1, & \mu = 0;\ \lambda \in \mathbb{C}\backslash\{0\} \\ \lambda\ldots(\lambda+n-1), & \mu = n \in \mathbb{N};\ \lambda \in \mathbb{C} \end{cases},$$

and the upper and lower incomplete gamma functions respectively [30,48],

$$\gamma(s,x) + \Gamma(s,x) = \int_0^x t^{s-1}\mathrm{e}^{-t}\mathrm{d}t + \int_x^\infty t^{s-1}\mathrm{e}^{-t}\mathrm{d}t = \Gamma(s),$$

considered for $x \ge 0$, $\Re(s) > 0$. Accordingly, the lower and upper incomplete Pochhammer symbols are defined as

$$(a;x)_\nu := \frac{\gamma(a+\nu,x)}{\Gamma(a)}, \qquad [a;x]_\nu := \frac{\Gamma(a+\nu,x)}{\Gamma(a)},$$

and satisfy the decomposition

$$(a;x)_\nu + [a;x]_\nu = (a)_\nu, \qquad a,\nu \in \mathbb{C},\ x \ge 0.$$

2.2 Generalized Fox–Wright and hypergeometric functions

The series definition of the Fox–Wright function is (cf. [13], [50, p. 4, Eq. (2.4)]):

$$
{}_p\Psi_q\left[\begin{matrix}(\mathbf{a}_p,\mathbf{A}_p)\\(\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\Big|z\right] = {}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\Big|z\right] = \sum_{k\geq 0}\frac{\prod_{j=1}^{p}\Gamma(a_j+kA_j)}{\prod_{j=1}^{q}\Gamma(b_j+kB_j)}\frac{z^k}{k!}, \qquad (2.1)
$$

where $A_j \geq 0,\ j=1,\ldots,p; B_m \geq 0,$ and $m=1,\ldots,q$. The series converges in the whole complex z-plane when

$$
\Delta = 1+\sum_{m=1}^{q}B_m-\sum_{j=1}^{p}A_j>0.
$$

If $\Delta=0$, then the series in (2.1) converges for $|z|<\rho$, and for $|z|=\rho$ under the condition $\Re(\mu)>\frac{1}{2}$, where

$$
\rho=\left(\prod_{j=1}^{p}A_j^{-A_j}\right)\left(\prod_{m=1}^{q}B_m^{B_m}\right)
$$

and

$$
\mu=\sum_{m=1}^{q}b_m-\sum_{j=1}^{p}a_j+\frac{p-q}{2}.
$$

The generalized hypergeometric function has the series representation [34, p. 404, §16.2]

$$
{}_pF_q\left[\begin{matrix}\mathbf{a}_p\\\mathbf{b}_q\end{matrix}\Big|z\right]=\sum_{k\geq 0}\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\frac{z^k}{k!}. \qquad (2.2)
$$

When $p\leq q$ the series converges for all finite values of z; when $p=q+1$ the series reduces to a hypergeometric polynomial of $\deg({}_{q+1}F_q)=q+1$. Moreover if none of a_j is non-positive integer, the series converges in the open disk $|z|<1$. On the circle $|z|=1$, the series converges provided

$$
\Re(\nabla)>-1, \quad \text{where} \quad \nabla=1+\sum_{m=1}^{q}b_m-\sum_{j=1}^{p}a_j
$$

except at $z=1$ if $\Re(\nabla)\in(0,1]$. In the special case $A_r=B_s=1$, the Fox–Wright function ${}_p\Psi_q[z]$ reduces (up to the multiplicative constant) to the generalized hypergeometric function

$$
{}_p\Psi_q\left[\begin{matrix}(\mathbf{a}_p,\mathbf{1})\\(\mathbf{b}_q,\mathbf{1})\end{matrix}\Big|z\right]=\frac{\Gamma(a_1)\ldots\Gamma(a_p)}{\Gamma(b_1)\ldots\Gamma(b_q)}\,{}_pF_q\left[\begin{matrix}\mathbf{a}_p\\\mathbf{b}_q\end{matrix}\Big|z\right].
$$

The case $p=2, q=1$ of (2.2) results in the Gaussian hypergeometric function

$$
{}_2F_1\left[\begin{matrix}a,b\\c\end{matrix}\Big|z\right]=\sum_{n\geq 0}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},
$$

whilst the confluence principle, which for fixed z gives

$$
\lim_{b\to\infty}{}_2F_1\left[\begin{matrix}a,b\\c\end{matrix}\Big|\frac{z}{b}\right]={}_1F_1[a,c,z]=\sum_{n\geq 0}\frac{(a)_n}{(c)_n}\frac{z^n}{n!},
$$

and this is the so-called confluent hypergeometric function (or Kummer function in other words).

Applying the lower incomplete Pochhammer symbol we introduce the lower incomplete Fox–Wright function [14, p. 5],

$$
{}_p\Psi_q^{(\gamma)}\left[\begin{matrix}(\mu,M,x),(\mathbf{a}_{p-1},\mathbf{A}_{p-1})\\(\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\Big|z\right] = \sum_{n\geq 0}\frac{\gamma(\mu+nM,x)\prod_{j=1}^{p-1}\Gamma(a_j+nA_j)}{\prod_{j=1}^{q}\Gamma(b_j+nB_j)}\frac{z^n}{n!}.
$$

Here the parameters $M, A_j, B_m > 0$ should satisfy the constraint

$$
\Delta^{(\gamma)}=1+\sum_{m=1}^{q}B_m-M-\sum_{j=1}^{p-1}A_j\geq 0.
$$

2.3 Horn's confluent Φ_2 and Φ_3 functions

The Horn confluent hypergeometric function of two variables is known also under Humbert's name (signified by the same symbols Φ_2, Φ_3, see [17]).

The definitions by Horn [16] were made precise and corrected by his student Borngässer [6]. The Horn's function [47, p. 25, Eq. (17)] is

$$
\Phi_2(b,b';c;x,y)=\sum_{m,n\geq 0}\frac{(b)_m(b')_n}{(c)_{m+n}}\frac{x^m\,y^n}{m!\,n!},
$$

whilst the other is defined by the series [8, p. 3]

$$
\Phi_3(\alpha;\beta;z,w)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha)_k}{(\beta)_{k+m}}\frac{z^k}{k!}\frac{w^m}{m!}.
$$

Both double series converge for all $\max(|z|,|w|)<\infty$.

2.4 The Srivastava–Daoust $\mathcal{S}$ function

Srivastava and Daoust generalized the Lauricella, Exton and Kampé de Fériet hypergeometric functions by the n-tuple power series [45, p. 454]

$$\mathcal{S}^{A:B^{(1)};\dots;B^{(n)}}_{C:D^{(1)};\dots;D^{(n)}}\left(\begin{matrix}[(a):\theta^{(1)},\dots,\theta^{(n)}]:[(b^{(1)}):\varphi^{(1)}];\dots;[(b^{(n)}):\varphi^{(n)}]\\ [(c):\psi^{(1)};\dots;\psi^{(n)}]:[(d^{(1)}):\delta^{(1)}];\dots;[(d^{(n)}):\delta^{(n)}]\end{matrix}\middle|\begin{matrix}x_1\\ \vdots\\ x_n\end{matrix}\right)$$

$$=\sum_{\boldsymbol{m}\geq 0}\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+\dots+m_n\theta_j^{(n)}}\times\prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{m_1\varphi_j^{(1)}}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\varphi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j^{(1)}+\dots+m_n\psi_j^{(n)}}\times\prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{m_1\delta_j^{(1)}}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!},$$

where $\boldsymbol{m} := (m_1,\dots,m_n)$ and the parameters satisfy

$$\theta_1^{(1)},\dots,\theta_A^{(1)},\dots,\delta_1^{(n)},\dots,\delta_{D^{(n)}}^{(n)} > 0.$$

For convenience, we write (a) to denote the sequence of A parameters $a_1,\dots,a_A$, with similar interpretations for $(b'),(b^{(1)}),\dots,(d^{(n)})$. Empty products should be interpreted as unity.

Srivastava and Daoust [46, pp. 157–158] reported that the series converges absolutely for all $x_1,\dots,x_n\in\mathbb{C}$, $\ell=1,\dots,n$,

$$\Delta_\ell = 1+\sum_{j=1}^{C}\psi_j^{(\ell)}+\sum_{j=1}^{D^{(\ell)}}\delta_j^{(\ell)}-\sum_{j=1}^{A}\theta_j^{(\ell)}-\sum_{j=1}^{B^{(\ell)}}\varphi_j^{(\ell)}>0.$$

When $\Delta_\ell=0$, $\ell=1,\dots,n$, the series converges inside discs $|x_\ell|<\eta_\ell$, where

$$\eta_\ell=\min_{\mu_1,\dots,\mu_n>0}\left\{\mu_\ell^{1+\sum_{j=1}^{D^{(\ell)}}\delta_j^{(\ell)}-\sum_{j=1}^{B^{(\ell)}}\varphi_j^{(\ell)}}\cdot\frac{\prod_{j=1}^{C}\left(\sum_{\ell=1}^{n}\mu_\ell\psi_j^{(\ell)}\right)^{\psi_j^{(\ell)}}\prod_{j=1}^{D^{(\ell)}}\left(\delta_j^{(\ell)}\right)^{\delta_j^{(\ell)}}}{\prod_{j=1}^{A}\left(\sum_{\ell=1}^{n}\mu_\ell\theta_j^{(\ell)}\right)^{\theta_j^{(\ell)}}\prod_{j=1}^{B^{(\ell)}}\left(\varphi_j^{(\ell)}\right)^{\varphi_j^{(\ell)}}}\right\}.$$

In turn, when all $\Delta_\ell<0$, $\mathcal{S}^{A:B^{(1)};\dots;B^{(n)}}_{C:D^{(1)};\dots;D^{(n)}}(x_1,\dots,x_n)$ diverges except at the origin, that is, this series is formal.

2.5 Exton's hypergeometric X function

Exton's double hypergeometric X-function [12]

$$X^{A:B;B'}_{C:D;D'}\left[\begin{matrix}(a):(b);(b')\\ (c):(d);(d')\end{matrix}\middle|x,\,y\right]=\sum_{k,n\geq 0}\frac{((a))_{2k+n}((b))_k((b'))_n}{((c))_{2k+n}((d))_k((d'))_n}\frac{x^k}{k!}\frac{y^n}{n!},$$

where (a) denotes the sequence of parameters $a_1,\dots,a_A$, whilst $((a))_m$ stands for the product $(a_1)_m\dots(a_A)_m$ and the empty product equals 1. Obviously, Exton's X function is a special case of the Srivastava–Daoust function in two variables:

$$X^{A:B;B'}_{C:D;D'}\left[\begin{matrix}a:b;b'\\ c:d;d'\end{matrix}\middle|x,y\right]=\mathcal{S}^{A:B;B'}_{C:D;D'}\left(\begin{matrix}[a:2;1]:[b:1];[b':1]\\ [c:2;1]:[d:1];[d':1]\end{matrix}\middle|\begin{matrix}x\\ y\end{matrix}\right).$$

Accordingly, the convergence conditions [46, pp. 157–158] of the $\mathcal{S}$ function reduce to

$$\begin{matrix}\Delta_1=1+2C+D-2A-B>0\\ \Delta_2=1+C+D'-A-B'>0\end{matrix},$$

under which Exton's double X series converges absolutely for all $x,y\in\mathbb{C}$. Other cases of convergence inside disks in $\mathbb{C}$ can be deduced from [46, pp. 153–157] in a straightforward way.

2.6 Kampé de Fériet hypergeometric function of two variables

The Kampé de Fériet generalized hypergeometric function of two variables is defined by the double series [47, p. 27, Eq. (28)]

$$F^{\,p:q;k}_{\,\ell:m;n}\left[\begin{matrix}(a_p):(b_q);(c_k)\\ (\alpha_\ell):(\beta_m);(\gamma_n)\end{matrix}\middle|x,y\right]=\sum_{r,s\geq 0}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{\ell}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!},$$

which converges when [46]

(i) $p+q<\ell+m+1$, $p+k<\ell+n+1$; $\max\{|x|,|y|\}<\infty$, or

(ii) $p+q=\ell+m+1$, $p+k=\ell+n+1$; and

$$\begin{cases} |x|^{\frac{1}{p-\ell}}+|y|^{\frac{1}{p-\ell}}<1, & \ell<p \\ \max(|x|,|y|)<1, & \ell>p. \end{cases}$$

The Kampé de Fériet function is also a special case of the Srivastava–Daoust $\mathscr{S}$ function.

2.7 ILHI and GL fractional derivative

The incomplete Lipschitz–Hankel integral (ILHI) of the first kind modified Bessel functions is defined by [1, p. 47, Eq. (5.41)]

$$I_{e_{\mu,\nu}}(z;a,b)=\int_0^z \mathrm{e}^{-bt}t^{\mu}I_\nu(at)\,\mathrm{d}t,$$

where $a,b>0$, $z,\nu,\mu\in\mathbb{C}$ and by convergence reasons $\Re(\mu+\nu)>-1$ (for more details on ILHI we can consult [1,10,11,27,31,43] and the references therein). The special case $I_{e_{\nu,\nu}}(x;a,b)$ takes place in (1.2). Obviously, $I_{e_{\mu,\nu}}(z;a,b)$ is the Laplace–Mellin transform of $I_\nu(at)\cdot \mathbb{1}_{[0,x]}(t)$, where $\mathbb{1}_S(t)$ denotes the indicator of the event $t\in S$.

The Grünwald–Letnikov fractional derivative (GLFC) of the order η with respect to the argument x of a suitable function f is defined by [44]

$$\mathbb{D}_x^\eta(f)=\lim_{h\to 0^+}\frac{1}{h^\eta}\sum_{m\geq 0}(-1)^m\binom{\eta}{m}f\left(x+(\eta-m)h\right).$$

As it is well-known (see, for example, [35]) the Grünwald–Letnikov fractional derivative of the order η of the exponential function

$$\mathbb{D}_x^\eta(\mathrm{e}^{\alpha x})=\alpha^\eta\mathrm{e}^{\alpha x},\qquad \alpha\in\mathbb{C}. \tag{2.3}$$

2.8 The cumulative distribution function

Any rv ξ defined on a standard probability space $(\Omega,\mathscr{A},\mathsf{P})$ has unique cdf

$$F_\xi(x)=\mathsf{P}(\xi<x),\qquad x\in\mathbb{R},$$

which is

(a) monotone non-decreasing;
(b) left-continuous;
(c) $F_\xi(-\infty)=0$; $F_\xi(\infty)=1$.

The reverse statement is also true, namely, any real function $F_\xi:\mathbb{R}\mapsto\mathbb{R}$ which satisfies (a)–(c) is a cdf of certain rv ξ.

The connection to pdf reads

$$F_\xi(x)=\int_{-\infty}^x f_\xi(t)\,\mathrm{d}t,\qquad x\in\mathbb{R};$$

or, if the cdf is absolutely continuous with respect to the Lebesgue measure $f_\xi(x)=F'_\xi(x)$, that is, the pdf equals to the Radon–Nikodým derivative of the cdf.

The pdf is non-negative and normalized.

Lemma 1 ([5, p. 45, 2.1.7]). *Let $F(x)$ be a* cdf *and $h>0$. Then*

$$H_1(x)=\frac{1}{h}\int_x^{x+h}F(t)\,\mathrm{d}t,\quad H_2(x)=\frac{1}{2h}\int_{x-h}^{x+h}F(t)\,\mathrm{d}t,$$

are also cdf*s.*

Lemma 2 ([5, p. 45, 2.1.8]). *Let $F(x)$ be a* cdf *of a continuous rv with $F(0)=0$. Then*

$$G(x)=\begin{cases}F(x)-F(x^{-1}), & x\geq 1\\ 0, & x<1\end{cases}$$

is also a cdf.

Hint for the proof: when $x\geq 1$,

$$\mathsf{P}\left\{\max\left(\xi,\frac{1}{\xi}\right)<x\right\}=\mathsf{P}\left\{\frac{1}{x}<\xi<x\right\}=F(x)-F(x^{-1}).$$

Lemma 3 ([37, p. 8, Eq. (20)]). *Let $r\in\mathbb{N}$. The* cdf *$H_r^{(1)}(x;h)$ we build by r-tuple successive application of the integral operator $H_1^{(1)}$ to the baseline* cdf *F_ξ. This gives*

$$H_r^{(1)}=\underbrace{H_1^{(1)}\circ\cdots\circ H_1^{(1)}}_{r}(F_\xi),$$

where $u\circ v$ means the composition of u,v. Then $H_r^{(1)}(x;h)$ is also a cumulative distribution function.

The positivity of rvs X,Y imply that $X+Y$ is *stochastically larger* than X. Therefore, we have the following.

Lemma 4 ([49, p. 1169, Lemma 1]). *Let X,Y be non-negative independent* rvs *and let $g:\mathbb{R}_+\to\mathbb{R}_+$ be nondecreasing. If f_X, f_{X+Y} are the* pdfs *of $X, X+Y$, respectively and F_Y is the* cdf *of Y, which may not have a* pdf *if F_Y is discontinuous. Then,*

$$\int_x^\infty g(t)\,f_{X+Y}(t)\,\mathrm{d}t>\int_x^\infty g(t)\,f_X(t)\,\mathrm{d}t,\quad x>0, \tag{2.4}$$

provided $F_Y(0)<1$ and the integrals exist.

3. The cdf F_I in terms of the incomplete Fox–Wright function

Our aim is to express cdf (1.2) in terms of the incomplete Fox–Wright function.

Theorem 1 ([14, p. 5, Theorem 1]). *For all $x \geq 0$, $\nu > -1/2$ and $b > a > 0$ there holds*

$$F_I(x;a,b;\nu) = \frac{a^\nu C_\nu}{2^\nu b^{2\nu+1}}\, {}_1\Psi_1^{(\gamma)}\left[\begin{array}{c}(2\nu+1,2,bx)\\ (\nu+1,1)\end{array}\,\middle|\,\frac{a^2}{4b^2}\right] \tag{3.1}$$

$$= \frac{a^\nu C_\nu}{2^\nu b^{2\nu+1}}\left\{2\,{}_1\Psi_1^{(\gamma)}\left[\begin{array}{c}(2\nu,2,bx)\\ (\nu,1)\end{array}\,\middle|\,\frac{a^2}{4b^2}\right] - \left(\frac{2b^2x}{a}\right)^\nu \mathrm{e}^{-bx} I_\nu(ax)\right\}. \tag{3.2}$$

Proof. Substituting $bt \mapsto u$ in (1.2) and using series form of I_ν in the integrand we get the series in terms of the lower incomplete gamma function:

$$F_I(x;a,b;\nu) = \frac{a^\nu C_\nu}{2^\nu b^{2\nu+1}} \sum_{n\geq 0} \frac{\left(\frac{a}{2b}\right)^{2n}\int_0^{bx} \mathrm{e}^{-u}u^{2(n+\nu)}\,\mathrm{d}u}{\Gamma(\nu+n+1)\,n!}$$

$$= \frac{a^\nu C_\nu}{2^\nu b^{2\nu+1}} \sum_{n\geq 0} \frac{\gamma(2\nu+1+2n,bx)\left(\frac{a^2}{4b^2}\right)^n}{\Gamma(\nu+n+1)\,n!},$$

which is the first equality stated in the theorem.

The second claim follows by the contiguous recurrence formula [34, p. 178, Eq. **8.8.1**]

$$\gamma(a+1,x) = a\,\gamma(a,x) - x^a\,\mathrm{e}^{-x},$$

which gives

$$F_I(x;a,b;\nu) = \frac{a^\nu C_\nu}{2^\nu b^{2\nu+1}}\left(\sum_{n\geq 0}\frac{\gamma(2\nu+2n,bx)}{\Gamma(\nu+n)\,n!}\left(\frac{a}{2b}\right)^{2n} - \frac{2(bx)^{2\nu}\mathrm{e}^{-bx}}{2\Gamma(\nu+1)}\,{}_0F_1\left[\begin{array}{c}-\\ \nu+1\end{array}\,\middle|\,\frac{(ax)^2}{4}\right]\right).$$

Having in mind that [34, p. 255, Eq. **10.39.9**]

$${}_0F_1\left[\begin{array}{c}-\\ \nu+1\end{array}\,\middle|\,\frac{z^2}{4}\right] = \Gamma(\nu+1)\left(\frac{2}{z}\right)^\nu I_\nu(z), \qquad -\nu\notin\mathbb{N},$$

we complete the proof of the second assertion.

Here we report on an uniform bound for ${}_1\Psi_1^{(\gamma)}$ and a functional bound for ${}_1F_2$.

Theorem 2 ([37, p. 3, Theorem 1.]). *For all $b > a > 0$, $\nu > -1/2$ and $x \geq 0$ we have*

$${}_1\Psi_1^{(\gamma)}\left[\begin{array}{c}(2\nu+1,2,bx)\\ (\nu+1,1)\end{array}\,\middle|\,\frac{a^2}{4b^2}\right] \leq \frac{4^\nu\left(\frac12\right)_\nu b^{2\nu+1}}{(b^2-a^2)^{\nu+\frac12}}.$$

Moreover, for $\nu \geq 0$, $a \geq 1$ and for all $x \geq 0$ there holds

$${}_1\Psi_1^{(\gamma)}\left[\begin{array}{c}(2\nu+1,2,bx)\\ (\nu+1,1)\end{array}\,\middle|\,\frac{a^2}{4b^2}\right] \leq \frac{b^{2\nu}x^{2\nu}(1-\mathrm{e}^{-bx})}{(2\nu+1)\,\Gamma(\nu+1)}\,{}_1F_2\left[\begin{array}{c}\nu+\frac12\\ \nu+1,\ \nu+\frac32\end{array}\,\middle|\,\frac{a^2x^2}{4}\right]. \tag{3.3}$$

Proof. According to (3.1), as $0 \leq F_I(x;a,b;\nu) \leq 1$, we have the uniform bound claim. As to the functional upper bound in (3.3), we apply the estimate [34, Eq. 8.10.2]

$$\gamma(a,t) \leq \frac{t^{a-1}}{a}\left(1-\mathrm{e}^{-t}\right), \qquad a\geq 1,\ t>0. \tag{3.4}$$

This bound, taken from (3.1) for $a = 2\nu+1+2n$ and $t = bx$, increases the sum and implies

$${}_1\Psi_1^{(\gamma)}\left[\begin{array}{c}(2\nu+1,2,bx)\\ (\nu+1,1)\end{array}\,\middle|\,\frac{a^2}{4b^2}\right]$$

$$\leq \sum_{n\geq 0}\frac{(bx)^{2(\nu+n)}\left(1-\mathrm{e}^{-bx}\right)}{4^n(2\nu+1+2n)\Gamma(\nu+1+n)\,n!}\left(\frac{a^2}{b^2}\right)^n$$

$$= \frac{(bx)^{2\nu}\left(1-\mathrm{e}^{-bx}\right)}{2\,\Gamma(\nu+1)}\sum_{n\geq 0}\frac{4^{-n}\Gamma\left(\nu+\frac12+n\right)(ax)^{2n}}{(\nu+1)_n\,\Gamma\left(\nu+\frac32+n\right)n!}$$

$$= \frac{(bx)^{2\nu}\left(1-\mathrm{e}^{-bx}\right)}{(2\nu+1)\Gamma(\nu+1)}\sum_{n\geq 0}\frac{\left(\nu+\frac12\right)_n\left(\frac{ax}{2}\right)^{2n}}{(\nu+1)_n\left(\nu+\frac32\right)_n n!},$$

which is equivalent to (3.3); in (3.4), $1 \leq a = 2\nu+1+2n$ holds for all $n\in\mathbb{N}_0$.

Theorem 3 ([37, p. 4, Theorem 2.]). *Let $b > a > 0$ and $2\nu+1 > 0$. Then for all $x \geq 1$ there holds the two-sided functional inequality*

$${}_1\Psi_1^{(\gamma)}\left[\frac{b}{x}\right] \leq {}_1\Psi_1^{(\gamma)}\left[bx\right] \leq {}_1\Psi_1^{(\gamma)}\left[\frac{b}{x}\right] + \frac{2\Gamma(2\nu)\,b^{2\nu+1}}{\Gamma(\nu)\,(b^2-a^2)^{\nu+\frac12}}, \tag{3.5}$$

where

$${}_1\Psi_1^{(\gamma)}[z] := {}_1\Psi_1^{(\gamma)}\left[\begin{array}{c}(2\nu+1,2,z)\\ (\nu+1,1)\end{array}\,\middle|\,\frac{a^2}{4b^2}\right].$$

Proof. Lemma 2 gives for $\xi \sim \mathrm{McKayI}(a,b,\nu)$, for all $x \geq 1$:

$$0 \leq G_I(x) = F_I(x;a,b;\nu) - F_I(x^{-1};a,b;\nu) \leq 1. \tag{3.6}$$

Hence,

$$ {}_1\Psi_1^{(\gamma)}\left[bx\right] - {}_1\Psi_1^{(\gamma)}\left[\frac{b}{x}\right] \geq 0, $$

which implies the left-hand-side inequality in (3.5).

Next, from $G_I(x) \leq 1$ we conclude

$$ {}_1\Psi_1^{(\gamma)}\left[bx\right] \leq {}_1\Psi_1^{(\gamma)}\left[\frac{b}{x}\right] + \frac{2^{2\nu} b^{2\nu+1}\Gamma\big(\nu+\frac{1}{2}\big)}{\sqrt{\pi}\,(b^2-a^2)^{\nu+\frac{1}{2}}} \tag{3.7} $$

$$ = {}_1\Psi_1^{(\gamma)}\left[\frac{b}{x}\right] + \frac{2\, b^{2\nu+1}\Gamma(2\nu)}{\Gamma(\nu)(b^2-a^2)^{\nu+\frac{1}{2}}}, \tag{3.8} $$

where (3.7) is obtained by the Legendre duplication formula for the gamma function. This explains at the same time that the quotient of gamma functions is well defined for the non-positive values of $\nu \in (-\frac{1}{2}, 0]$ in (3.8). The rest is obvious.

Theorem 4 ([37, p. 5, Theorem 3.]). *For all $b > a > 0$, $\nu > -1/2$ and for all $x \geq 0$ it holds true that*

$$ \frac{1}{2}\left(\frac{2b^2x}{a}\right)^{\nu} \mathrm{e}^{-bx} I_\nu(ax) \leq {}_1\Psi_1^{(\gamma)}\left[\begin{array}{c} (2\nu, 2, bx) \\ (\nu, 1) \end{array}\Bigg|\, \frac{a^2}{4b^2}\right] $$

$$ \leq \frac{1}{2}\left(\frac{2b^2x}{a}\right)^{\nu} \mathrm{e}^{-bx} I_\nu(ax) + \frac{2^{2\nu-1} b^{2\nu+1}\,\big(\frac{1}{2}\big)_\nu}{(b^2-a^2)^{\nu+\frac{1}{2}}}. $$

Moreover, when $x \geq 1$, we have the functional inequality

$$ 0 \leq {}_1\Psi_1^{(\gamma)}\left[bx\right] - {}_1\Psi_1^{(\gamma)}\left[\frac{b}{x}\right] - H(x) \leq \frac{\Gamma(2\nu)\, b^{2\nu+1}}{\Gamma(\nu)\,(b^2-a^2)^{\nu+\frac{1}{2}}}, \tag{3.9} $$

where

$$ H(x) := \frac{1}{2}\left(\frac{2b^2}{a}\right)^{\nu}\left\{x^\nu \mathrm{e}^{-bx}\, I_\nu(ax) - x^{-\nu}\mathrm{e}^{-\frac{b}{x}} I_\nu\left(\frac{a}{x}\right)\right\}, $$

and ${}_1\Psi_1^{(\gamma)}[z]$ remains the same as in Theorem 3.

Proof. The rv McKayI(a, b, ν) generates a counterpart result to the representation formula (3.2) in Theorem 1 for the related cdf, also in terms of ${}_1\Psi_1^{(\gamma)}$, the exponential function and I_ν. We apply the standard bilateral bound $0 \leq F_I(x; a, b; \nu) \leq 1$ to (3.2) to infer the first inequality, in turn, the application of Lemma 2 results in the bilateral bound (3.9). The detailed proving procedure is therefore omitted.

Now, we present the results obtained by means of Lemmas 1 and 3.

Theorem 5 ([37, p. 6, Theorem 4.]). *Let the* rv *$X \sim$* McKay I *(a, b, ν), and the* cdf *$F_{I,1}^{(1)}(x; h)$ be defined by* Lemma 1 *with the baseline cdf $F_I(x)$. Then for all $b > a > 0$, $\nu > -\frac{1}{2}$ and $x, h \geq 0$ we have*

$$ \begin{aligned} F_{I,1}^{(1)}(x;h) &= \frac{\sqrt{\pi}\,(b^2-a^2)^{\nu+\frac{1}{2}}}{4^\nu b^{2\nu+1}\, h\Gamma\big(\nu+\frac{1}{2}\big)}\Bigg\{(x+h) \\ &\cdot {}_2\Psi_2^{(\gamma)}\left[\begin{array}{c} (2\nu+2, 2, b(x+h)),\ \big(\nu+\frac{1}{2}, 1\big) \\ (\nu+1, 1),\ \big(\nu+\frac{3}{2}, 1\big) \end{array}\Bigg|\, \frac{a^2}{4b^2}\right] \\ &- \frac{2}{b}\,{}_1\Psi_1^{(\gamma)}\left[\begin{array}{c} (2\nu+2, 2, b(x+h)) \\ (\nu+1, 1) \end{array}\Bigg|\, \frac{a^2}{4b^2}\right] \\ &+ \frac{2}{b}\,{}_1\Psi_1^{(\gamma)}\left[\begin{array}{c} (2\nu+2, 2, bx) \\ (\nu+1, 1) \end{array}\Bigg|\, \frac{a^2}{4b^2}\right] \\ &+ \frac{2[b(x+h)]^{2\nu+2}\mathrm{e}^{-b(x+h)}}{b\,(2\nu+1)\Gamma(\nu+1)} \\ &\cdot {}_1F_2\left[\begin{array}{c} \nu+\frac{1}{2} \\ \nu+1,\ \nu+\frac{3}{2} \end{array}\Bigg|\, \frac{a^2}{4}(x+h)^2\right] \\ &- x\,{}_2\Psi_2^{(\gamma)}\left[\begin{array}{c} (2\nu+2, 2, bx),\ \big(\nu+\frac{1}{2}, 1\big) \\ (\nu+1, 1),\ \big(\nu+\frac{3}{2}, 1\big) \end{array}\Bigg|\, \frac{a^2}{4b^2}\right] \\ &- \frac{2(bx)^{2\nu+2}\mathrm{e}^{-bx}}{b\,(2\nu+1)\Gamma(\nu+1)}\,{}_1F_2\left[\begin{array}{c} \nu+\frac{1}{2} \\ \nu+1,\ \nu+\frac{3}{2} \end{array}\Bigg|\, \frac{a^2}{4}x^2\right]\Bigg\}. \end{aligned} $$

Corollary 1 ([37, p. 8, Corollary 1.]). *Let the* cdf *$F_{I,1}^{(2)}(x; h)$ be defined by* Lemma 1*, second case. Then for all $b > a > 0$, $\nu > -\frac{1}{2}$ and $x \geq 0$; $h > 0$ we have*

$$ \begin{aligned} F_{I,1}^{(2)}(x;h) &= \frac{C_\nu}{h}\Bigg\{\frac{x+h}{2} \\ &\cdot {}_2\Psi_2^{(\gamma)}\left[\begin{array}{c} (2\nu+2, 2, b(x+h)),\ \big(\nu+\frac{1}{2}, 1\big) \\ (\nu+1, 1),\ \big(\nu+\frac{3}{2}, 1\big) \end{array}\Bigg|\, \frac{a^2}{4b^2}\right] \\ &- \frac{1}{b}\,{}_1\Psi_1^{(\gamma)}\left[\begin{array}{c} (2\nu+2, 2, b(x+h)) \\ (\nu+1, 1) \end{array}\Bigg|\, \frac{a^2}{4b^2}\right] \\ &+ \frac{(b(x+h))^{2\nu+2}\mathrm{e}^{-b(x+h)}}{b\,(2\nu+1)\Gamma(\nu+1)}\,{}_1F_2\left[\begin{array}{c} \nu+\frac{1}{2} \\ \nu+1,\ \nu+\frac{3}{2} \end{array}\Bigg|\, \frac{a^2}{4}(x+h)^2\right] \\ &- \frac{x-h}{2}\,{}_2\Psi_2^{(\gamma)}\left[\begin{array}{c} (2\nu+2, 2, b(x-h)),\ \big(\nu+\frac{1}{2}, 1\big) \\ (\nu+1, 1),\ \big(\nu+\frac{3}{2}, 1\big) \end{array}\Bigg|\, \frac{a^2}{4b^2}\right] \\ &+ \frac{1}{b}\,{}_1\Psi_1^{(\gamma)}\left[\begin{array}{c} (2\nu+2, 2, b(x-h)) \\ (\nu+1, 1) \end{array}\Bigg|\, \frac{a^2}{4b^2}\right] \\ &- \frac{(b(x-h))^{2\nu+2}\mathrm{e}^{-b(x-h)}}{b\,(2\nu+1)\Gamma(\nu+1)} \\ &\cdot {}_1F_2\left[\begin{array}{c} \nu+\frac{1}{2} \\ \nu+1,\ \nu+\frac{3}{2} \end{array}\Bigg|\, \frac{a^2}{4}(x-h)^2\right]\Bigg\}. \end{aligned} $$

Theorem 6 ([37, p. 9, Theorem 5.]). *For all $b > a > 0$; $2\nu + 1 > 0$; $r \in \mathbb{N}$ and $x \geq 0$, $h > 0$ we have*

$$F_{I,r}^{(1)}(x;h) = \frac{C_\nu}{r!} \left(\frac{x}{h}\right)^r \sum_{k=0}^r \binom{r}{k} \frac{(-1)^{k+1}}{(b\,x)^k} \cdot \left\{ {}_1\Psi_1^{(\gamma)} \left[\begin{matrix} (2\nu+1+k, 2, bx) \\ (\nu+1, 1) \end{matrix} \Big| \frac{a^2}{4b^2} \right] - \left(1 + \frac{h}{x}\right)^{r-k} \times {}_1\Psi_1^{(\gamma)} \left[\begin{matrix} (2\nu+1+k, 2, b(x+h)) \\ (\nu+1, 1) \end{matrix} \Big| \frac{a^2}{4b^2} \right] \right\}.$$

Proof. Denote $\mathbf{dx}^{r-1} := \mathrm{d}x_1\, \mathrm{d}x_2 \ldots \mathrm{d}_{x_{r-1}}$, so that

$$F_{I,r}^{(1)}(x;h) = \frac{C_\nu}{h^r} \sum_{n\geq 0} \frac{1}{\Gamma(\nu+1+n)\, n!} \cdot \left(\frac{a^2}{4b^2}\right)^n \int_{[x,x+h]\times\prod_{j=1}^{r-1}[x_j,x_j+h]} \gamma(2\nu+1+2n, bt)\, \mathrm{d}t\, \mathbf{dx}^{r-1}.$$

Apply the notion

$$I(\alpha,\beta;x) = \int_{[0,x]\times\prod_{j=1}^{r-1}[0,x_j]} \gamma(\alpha,\beta t)\, \mathrm{d}t\, \mathbf{dx}^{r-1}.$$

The use of special form of the formula [41, p. 23, Eq. **1.2.1.1**] for $\min(\alpha,\beta) > 0$, $\lambda \geq 0$ implies

$$\int_0^x x^\lambda \gamma(\alpha,\beta x)\, \mathrm{d}x = \frac{x^{\lambda+1}}{\lambda+1}\gamma(\alpha,\beta x) - \frac{\gamma(\lambda+1+\alpha,\beta x)}{(\lambda+1)\,\beta^{\lambda+1}},$$

which provides

$$I(\alpha,\beta;x) = \frac{x^r}{r!}\sum_{k=0}^r \binom{r}{k} \frac{(-1)^k}{(\beta x)^k}\gamma(\alpha+k,\beta x),$$

and we obtain $I(\alpha,\beta;x)$ by induction. Hence,

$$F_{I,r}^{(1)}(x;h) = \frac{C_\nu}{h^r}\sum_{n\geq 0}\frac{1}{\Gamma(\nu+1+n)\, n!}\left(\frac{a^2}{4b^2}\right)^n \cdot \big[I(2\nu+1+2n, b; x+h) - I(2\nu+1+2n, b; x)\big]$$

and therefore

$$F_{I,r}^{(1)}(x;h) = \frac{C_\nu\, x^r}{r!\, h^r}\sum_{k=0}^r \binom{r}{k}\frac{(-1)^{k+1}}{(b\,x)^k} \cdot \left\{ \sum_{n\geq 0}\frac{\gamma(2\nu+1+k+2n, bx)}{\Gamma(\nu+1+n)\, n!}\left(\frac{a^2}{4b^2}\right)^n - \sum_{n\geq 0}\frac{\gamma(2\nu+1+k+2n, b(x+h))}{\Gamma(\nu+1+n)\, n!} \cdot \left(1+\frac{h}{x}\right)^{r-k}\left(\frac{a^2}{4b^2}\right)^n \right\}$$

$$= \frac{C_\nu}{r!}\left(\frac{x}{h}\right)^r \sum_{k=0}^r \binom{r}{k}\frac{(-1)^{k+1}}{(b\,x)^k} \cdot \left\{ {}_1\Psi_1^{(\gamma)}\left[\begin{matrix} (2\nu+1+k, 2, bx) \\ (\nu+1,1) \end{matrix} \Big| \frac{a^2}{4b^2}\right] - \left(1+\frac{h}{x}\right)^{r-k} {}_1\Psi_1^{(\gamma)} \times \left[\begin{matrix} (2\nu+1+k, 2, b(x+h)) \\ (\nu+1,1) \end{matrix} \Big| \frac{a^2}{4b^2}\right] \right\}.$$

In turn, this proves the statement of the theorem.

Remark 1 ([37, p. 8, Remark 2.]). The first type of two-sided inequalities which we can obtain are the straightforward consequences of $0 \leq F_{I,1}^{(j)}(x;h) \leq 1$; $j = 1, 2$ for the same parameter space $b > a > 0$; $2\nu+1 > 0$, $h > 0$ as in Corollary 1 and Theorem 6, respectively.

On the other hand mimicking (3.6) in Lemma 2, generating with the baseline cdfs $F_{I,1}^{(j)}(x;h)$ another associated cdfs $G_{I,1}^{(j)}(x;h)$; $j = 1, 2$, a new set of bilateral inequalities follow for $\mathrm{supp}(G_{I,1}^{(j)}) = [1,\infty)$; $j = 1, 2$ for positive $h > 0$. These results can also be understood as a kind of monotonicity with respect to the argument x since the cdfs are *a fortiori* monotone nondecreasing.

Corollary 2 ([37, p. 10, Corollary 2.]). *Let the situation be the same as in* Theorem 6. *Denote*

$$\mathscr{A}_r\left({}_1\Psi_1^{(\gamma)}; t\right) = \sum_{k=0}^r \frac{(-r)_k}{b^k\, k!} t^{r-k} \cdot {}_1\Psi_1^{(\gamma)}\left[\begin{matrix} (2\nu+1+k, 2, bt) \\ (\nu+1,1) \end{matrix} \Big| \frac{a^2}{4b^2}\right].$$

Then, for all $x > 0$ we have

$$0 \leq \mathscr{A}_r\left({}_1\Psi_1^{(\gamma)}; x+h\right) - \mathscr{A}_r\left({}_1\Psi_1^{(\gamma)}; x\right) \leq \frac{a^\nu\, C_\nu}{2^\nu\, b^{2\nu+1}}\, r!h^r.$$

Proof. The statement follows since $F_{I,r}^{(1)}(x;h)$ is a cdf having unit interval co-domain for the $\mathrm{supp}\big(F_{I,r}^{(1)}\big) = \mathbb{R}_+$.

4. F_I in terms of Exton's X function

4.1 ILHI *via* GLFD in terms of Exton X function

Here we express the incomplete Lipschitz–Hankel integral ILHI $I_{e_{\mu,\nu}}(x;b)$ with the aid of the Grünwald–Letnikov fractional derivative in terms of the Exton X function. Firstly, we recall the definition of ILHI:

$$I_{e_{\mu,\nu}}(z;a,b) = \int_0^z e^{-bt} t^\mu I_\nu(at)\,dt,$$

where $a, b > 0$, $z, \nu, \mu \in \mathbb{C}$ and $\Re(\mu+\nu) > -1$.

Theorem 7 ([14, p. 8, Theorem 3]). *For all positive a, b and complex $z, \nu, \mu \in \mathbb{C}$ that $\Re(\nu) > -1$, we have*

$$I_{e_{\mu,\nu}}(z;a,b) = (-1)^\mu \mathbb{D}_b^\mu \left(X^{1:0;0}_{1:1;0} \left[\begin{matrix} \nu+1 : -;- \\ \nu+2 : \nu+1;- \end{matrix} \middle| \frac{a^2z^2}{4}, -bz \right] \right). \tag{4.1}$$

Moreover, when additionally $\Re(\mu+\nu) > -1$, there holds

$$I_{e_{\mu,\nu}}(z;a,b) = \frac{\left(\frac{a}{2}\right)^\nu z^{\mu+\nu+1}}{(\mu+\nu+1)\Gamma(\nu+1)} \cdot X^{1:0;0}_{1:1;0} \left[\begin{matrix} \mu+\nu+1 : -;- \\ \mu+\nu+2 : \nu+1;- \end{matrix} \middle| \frac{a^2z^2}{4}, -bz \right]. \tag{4.2}$$

Proof. By means of (2.3) we reduce the Laplace–Mellin kernel in the integrand of ILHI:

$$I_{e_{\mu,\nu}}(z;a,b) = (-1)^\mu \int_0^z \mathbb{D}_b^\mu \left(e^{-bt}\right) I_\nu(at)dt = (-1)^\mu \mathbb{D}_b^\mu \left(\int_0^z e^{-bt} I_\nu(at)\,dt \right).$$

The Taylor series of both e^{-bt} and $I_\nu(at)$ imply

$$\int_0^z e^{-bt} I_\nu(at)\,dt = \frac{a^\nu z^{\nu+1}}{2^\nu\,\Gamma(\nu+2)} \sum_{k,j\ge 0} \frac{(\nu+1)_{2k+j} \left(\frac{az}{2}\right)^{2k+\nu} (-bz)^j}{(\nu+2)_{2k+j}(\nu+1)_k\, k!\, j!} = \frac{a^\nu z^{\nu+1}}{2^\nu \Gamma(\nu+2)} X^{1:0;0}_{1:1;0} \left[\begin{matrix} \nu+1 : -;- \\ \nu+2 : \nu+1;- \end{matrix} \middle| \frac{a^2z^2}{4}, -bz \right],$$

so is the proof of (4.1).

Now, expand into Taylor series the integrand of ILHI. Thus,

$$I_{e_{\mu,\nu}}(z;a,b) = \frac{\left(\frac{a}{2}\right)^\nu z^{\mu+\nu+1}}{(\mu+\nu+1)\Gamma(\nu+1)} \cdot \sum_{k,j\ge 0} \frac{(\mu+\nu+1)_{2k+j} \left(\frac{az}{2}\right)^{2k} (-bz)^j}{(\mu+\mu+2)_{2k+j}(\nu+1)_k\, k!\, j!}.$$

Exton's functions converge in $\mathbb{C}$ as $\Delta_1 = 2$, $\Delta_2 = 1$.

Corollary 3 ([14, p. 9, Corollary 1.]). *For all $b > a > 0$, $\nu > -1/2$ we have for any $x \ge 0$*

$$F_I(x;a,b;\nu) = C_\nu\, I_{e_{\nu,\nu}}(x;a,b),$$

where the cdf *can be built either with (4.1) or (4.2).*

Another fashion expression for the cdf F_I in terms of Exton's X function reads as follows.

Theorem 8 ([14, p. 10, Theorem 4.]). *For all ν non-negative integer, $b > a > 0$ and $x \ge 0$ there holds*

$$F_I(x;a,b;\nu) = \frac{(b^2-a^2)^{\nu+1/2}}{\Gamma(2\nu+1)} x^{2\nu+1} e^{-bx} \cdot X^{0:1;1}_{1:0;0} \left[\begin{matrix} - : \nu+\frac12;1 \\ 2\nu+2 : -;- \end{matrix} \middle| a^2x^2, bx \right]. \tag{4.3}$$

Proof. Since

$$\mathscr{I}_\nu = \int_0^x e^{-bt} t^\nu I_\nu(at)\,dt = \frac{a^\nu x^{2\nu+1}}{2^\nu(2\nu+1)\Gamma(\nu+1)} e^{-bx} \cdot \sum_{m\ge 0} \frac{\left(\nu+\frac12\right)_m \left(\frac{ax}{2}\right)^{2m}}{\left(\nu+\frac32\right)_m (\nu+1)_m\, m!}\, {}_1F_1\left[\begin{matrix} 1 \\ 2(m+\nu+1) \end{matrix} \middle| bx \right],$$

putting $2(\nu+m+1) \mapsto m$ in the identity [42, p. 488, Eq. 7.11.2.13] we get for all $m \in \mathbb{N}_0$

$${}_1F_1\left[\begin{matrix} 1 \\ 2(m+\nu+1) \end{matrix} \middle| bx \right] = \sum_{j\ge 0} \frac{\Gamma\big(2m+2\nu+2\big)\,(bx)^j}{(j+2m+2\nu+1)!},$$

which leads to the expression

$$\mathscr{I}_\nu = \frac{\left(\frac{a}{2}\right)^\nu x^{2\nu+1} e^{-bx}}{(2\nu+1)\Gamma(\nu+1)} \sum_{m,j\ge 0} \frac{\left(\nu+\frac12\right)_m (1)_j}{(2\nu+2)_{2m+j}} \frac{(ax)^{2m}}{m!} \frac{(bx)^j}{j!} = \frac{\left(\frac{a}{2}\right)^\nu x^{2\nu+1} e^{-bx}}{(2\nu+1)\Gamma(\nu+1)} X^{0:1;1}_{1:0;0} \left[\begin{matrix} - : \nu+\frac12;1 \\ 2\nu+2 : -;- \end{matrix} \middle| a^2x^2, bx \right].$$

This completes the proof of (4.3).

4.2 Bounds for Exton's X function

Theorem 9 ([21, p. 261, Theorem 1]). *Denote*

$$\mathfrak{H}_\nu(x;a,b) = \frac{(2\nu+1)_3 \gamma(2\nu+1, bx)}{a^2\, b^{2\nu+1}}\, {}_0F_1\left[-;\nu+1; \frac{(ax)^2}{4} \right].$$

Then for all $b > a > 0$, $\nu > -1/2$ it is

$$\mathfrak{H}_\nu(x;a,b) - \frac{\Gamma(2\nu+4)}{a^2(b^2-a^2)^{\nu+\frac12}} \le x^{2\nu+3} X^{1:0;1}_{1:1;1} \left[\begin{matrix} 2\nu+3 : \; - \; ; 2\nu+1 \\ 2\nu+4 : \nu+2 ; 2\nu+2 \end{matrix} \middle| \frac{(ax)^2}{4}, -bx \right] \le \mathfrak{H}_\nu(x;a,b), \qquad x \ge 0.$$

Moreover, for $\nu \in \mathbb{N}_0$ and $b > a > 0$, $x \ge 0$ there holds

$$x^{2\nu+1} X^{0:1;1}_{1:0;0} \left[\begin{matrix} - : \nu+\frac12;1 \\ 2\nu+2 : -;- \end{matrix} \middle| (ax)^2, bx \right] \le \frac{\Gamma(2\nu+1)\, e^{bx}}{(b^2-a^2)^{\nu+1/2}}.$$

Proof. Starting with the representation [19, p. 156, Corollary 6]

$$F_I(x;a,b;\nu) = \frac{(b^2-a^2)^{\nu+1/2}}{b^{2\nu+1}\Gamma(2\nu+1)} \cdot \left\{\gamma(2\nu+1,bx)\,{}_0F_1\left[-;\nu+1;\frac{(ax)^2}{4}\right] - \frac{a^2b^{2\nu+1}x^{2\nu+3}}{(2\nu+1)_3} \cdot X^{1:0;1}_{1:1;1}\left[\begin{matrix}2\nu+3:-;2\nu+1\\ 2\nu+4:\nu+2;2\nu+2\end{matrix}\middle|\frac{(ax)^2}{4},-bx\right]\right\},$$

we apply the bilateral inequality $0 \le F_I(x;a,b;\nu) \le 1$. Now, obvious steps lead to the asserted bounds.

Next, consider the expression for the cdf F_I reported for the parameters $\nu \in \mathbb{N}_0$, $b > a > 0$. Precisely, for all $x \ge 0$ there holds (4.3). Since $F_I(x;a,b;\nu) \le 1$ we conclude the second upper bound.

The following more sophisticated bilateral bound follows by applying the Lemma 4.

Theorem 10 ([21, p. 262, Theorem 2]). *For all $\min(\nu,\mu) > -1/2$, $b > a > 0$ and $x \ge 0$ we have*

$$\frac{(1-(a/b)^2)^{\mu+\frac12}}{\Gamma(2\mu+2\nu+2)}\gamma(2\mu+2\nu+2,bx)\,{}_0F_1\left[-;\mu+\nu+\frac32;\frac{(ax)^2}{4}\right] - \frac{\gamma(2\nu+1,bx)}{\Gamma(2\nu+1)}\,{}_0F_1\left[-;\nu+1;\frac{(ax)^2}{4}\right]$$
$$< a^2b^{2\nu+1}x^{2\nu+3}\left\{\frac{[(b^2-a^2)^{\mu+\frac12}x^{2\mu+1}}{\Gamma(2\mu+2\nu+5)}X^{1:0;1}_{1:1;1}\left[\mu+\nu+\tfrac12\right] - \frac{1}{\Gamma(2\nu+4)}X^{1:0;1}_{1:1;1}[\nu]\right\},$$

where the shorthand

$$X^{1:0;1}_{1:1;1}[\nu] = X^{1:0;1}_{1:1;1}\left[\begin{matrix}2\nu+3:-;2\nu+1\\ 2\nu+4:\nu+2;2\nu+2\end{matrix}\middle|\frac{(ax)^2}{4},-bx\right].$$

Firstly, it follows by obvious transformations the uniform bound for ILHI [21, p. 263]

$$I_{e_{\nu,\nu}}\left(ax;1,\frac{b}{a}\right) \le \frac{2^\nu\left(\frac12\right)_\nu}{[(b/a)^2-1]^{\nu+\frac12}}.$$

Next, having in mind the previously exposed Theorems 7 and 8, we conclude the next results. Denote in this goal

$$\mathscr{X}^{1:0;0}_{1:1;0}[z;\mu,\nu] = X^{1:0;0}_{1:1;0}\left[\begin{matrix}\mu+1:-;-\\ \mu+2:\nu+1;-\end{matrix}\middle|\frac{a^2z^2}{4},-bz\right].$$

Theorem 11 ([21, p. 263, Corollary 1.]). *Let $b > a > 0$, $\nu > -1/2$. Then for all $x \ge 0$ there hold the upper bounds*

$$\mathbb{D}^\nu_b\left(\mathscr{X}^{1:0;0}_{1:1;0}[x;\nu,\nu]\right) \le \frac{(-2a)^\nu\Gamma\left(\nu+\frac12\right)}{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}},$$

$$x^{2\nu+1}\,\mathscr{X}^{1:0;0}_{1:1;0}[x;2\nu,\nu] \le \frac{\Gamma(2\nu+2)}{(b^2-a^2)^{\nu+1/2}}.$$

5. F_I expressed by Horn's function Φ_2

In [19, p. 149, Theorem 3] the following representation formula is proved

Theorem 12. *For all $x \ge 0$, when $\nu > -1/2$ and $b > a > 0$ there holds*

$$F_I(x;a,b;\nu) = \frac{(b^2-a^2)^{\nu+1/2}}{\Gamma(2\nu+2)}x^{2\nu+1} \cdot \Phi_2\left(\nu+\frac12,\nu+\frac12;2\nu+2;(a-b)x,-(a+b)x\right). \quad (5.1)$$

The relation (5.1) implies the functional upper bound for the Horn confluent hypergeometric function of two variables Φ_2.

Theorem 13 ([22, p. 125, Proposition 2.1]). *For all $b > a > 0$; $\nu > -1/2$ and for all $x \ge 0$ there holds*

$$x^{2\nu+1}\,\Phi_2\left(\nu+\frac12,\nu+\frac12;2\nu+2;x(a-b),-x(b+a)\right) \le \frac{\Gamma(2\nu+2)}{(b^2-a^2)^{\nu+1/2}}.$$

Considering the baseline cdf $F_I(x;a,b;\nu)$ which is defined in Lemma 1, the cdf is

$$H_1(x) = \frac1h\int_x^{x+h}F_I(t)\,\mathrm{d}t = \frac{(b^2-a^2)^{\nu+1/2}}{\Gamma(2\nu+3)\,h}\left\{(x+h)^{2\nu+2}\,\Phi_2^{[2\nu+3]}(x+h) - x^{2\nu+2}\Phi_2^{[2\nu+3]}(x)\right\},$$

where the shorthand

$$\Phi_2^{[\eta]}(x) = \Phi_2(\eta,\eta;2\eta+1;(a-b)x,-(a+b)x),$$

we get the result by the bounds $0 \le H_1(x), H_2(x) \le 1$, respectively.

Theorem 14 (**[22, p. 126, Theorem 2.3.]**). *For all $x \geq 0$, $h > 0$ and $\nu > -\frac{1}{2}$, $b > a > 0$ there holds*

$$\frac{\Phi_2\big(\nu+\frac12,\nu+\frac12;2\nu+2;(a-b)x,-(a+b)x\big)}{\Phi_2\big(\nu+\frac12,\nu+\frac12;2\nu+2;(a-b)(x+h),-(a+b)(x+h)\big)} \leq \left(1+\frac{h}{x}\right)^{2\nu+2},$$

$$\frac{\Phi_2\big(\nu+\frac12,\nu+\frac12;2\nu+2;(a-b)(x-h),-(a+b)(x-h)\big)}{\Phi_2\big(\nu+\frac12,\nu+\frac12;2\nu+2;(a-b)(x+h),-(a+b)(x+h)\big)} \leq \left(\frac{x+h}{x-h}\right)^{2\nu+2}.$$

Bearing in mind Lemma 2 applied for the baseline cdf $F_I(x)$, we get the following bounds by the same derivation method.

Theorem 15 (**[22, p. 127, Theorem 2.5.]**). *For all $\nu > -\frac{1}{2}$, $b > a > 0$ and for all $x \geq 1$ we have*

$$\frac{\Phi_2\big(\nu+\frac12,\nu+\frac12;2\nu+2;(a-b)/x,-(a+b)/x\big)}{\Phi_2\big(\nu+\frac12,\nu+\frac12;2\nu+2;(a-b)x,-(a+b)x\big)} \leq x^{4\nu+2}.$$

Moreover, for the same domain and parameter space there holds true the uniform bound

$$x^{2\nu+1}\,\Phi_2\left(\nu+\frac12,\nu+\frac12;2\nu+2;(a-b)x,-(a+b)x\right) \times\left(\nu+\frac12,\nu+\frac12;2\nu+2;\frac{a-b}{x},-\frac{a+b}{x}\right) \leq \frac{\Gamma(2\nu+2)}{(b^2-a^2)^{\nu+\frac12}}.$$

5.1 F_I generated results by Lemma 4

Consider the stochastic subordination Lemma 4. With the help of this inequality we prove a strict monotonicity of the cdf (5.1) in Theorem 12 and certain consequences of monotone behaviour of F_I.

Theorem 16 (**[23, Theorem 2.1]**). *If $\min(\nu_1,\nu_2) > -1/2$, $x \geq 0$ and $b > a > 0$ there holds*

$$F_I(x;a,b;\nu_1+\nu_2+1/2) < F_I(x;a,b;\nu_1). \qquad (5.2)$$

Moreover, for the same parameter range, the following inequality follows

$$\frac{I_{\nu_1+\nu_2+1/2}(ax) \mp I_{\nu_1+\nu_2+3/2}(ax)}{I_{\nu_1}(ax) \mp I_{\nu_1+1}(ax)} < \frac{\Gamma(\nu_1+\nu_2+2)}{\Gamma(\nu_1+\frac32)}\left(\frac{2a}{(b^2-a^2)x}\right)^{\nu_2+1/2}$$

$$x^{2\nu_2+1}\,\frac{\Phi_2^{[\nu_1+\nu_2+1]}(x)}{\Phi_2^{[\nu_1+\frac12]}(x)} < \frac{\Gamma(2\nu_1+2\nu_2+3)}{(b^2-a^2)^{\nu_2+\frac12}\,\Gamma(2\nu_1+2)}.$$

Proof. The moment generating function (mgf) of the rv $\xi \sim$ McKay I$(a,b;\nu)$ equals

$$\begin{aligned} M_\xi(s) &= \mathsf{E}\,\mathrm{e}^{s\xi} = \int_0^\infty \mathrm{e}^{sx} f_I(x;a,b;\nu)\,\mathrm{d}x \\ &= \frac{\sqrt{\pi}(b^2-a^2)^{\nu+1/2}}{(2a)^\nu\Gamma\left(\nu+\frac12\right)}\int_0^\infty \mathrm{e}^{(s-b)x}x^\nu I_\nu(ax)\,\mathrm{d}x \\ &= \left(1-\frac{s(2b-s)}{b^2-a^2}\right)^{-\nu-\frac12}. \end{aligned}$$

Since $\xi \sim f_I(x;a,b;\nu_1)$ and $\upsilon \sim f_I(x;a,b;\nu_2)$ being independent, the mgf of $\xi+\upsilon$ is

$$M_{\xi+\upsilon}(s) = M_\xi(s)M_\upsilon(s) = \left(1-\frac{s(2b-s)}{b^2-a^2}\right)^{-\nu_1-\nu_2-1},$$

which implies that rv $\xi+\upsilon \sim f_I(x;a,b;\nu_1+\nu_2+1/2)$. Now, rewriting (2.4) in the form

$$\int_0^x g(t)f_{X+Y}(t)\,\mathrm{d}t < \int_0^x g(t)f_X(t)\,\mathrm{d}t, \qquad (5.3)$$

and taking $g(x)=1$ for all $x>0$, we conclude the first statement:

$$\int_0^x f_I(t;a,b;\nu_1+\nu_2+1/2)\,\mathrm{d}t < \int_0^x f_I(t;a,b;\nu_1)\,\mathrm{d}t.$$

Now, from (5.3) for $g(x)=\mathrm{e}^{(b\pm a)x}$ (Lemma 4), it follows

$$\frac{(b^2-a^2)^{\nu_2+1/2}\Gamma(\nu_1+1/2)}{(2a)^{\nu_2+1/2}\Gamma(\nu_1+\nu_2+1)}\int_0^x \mathrm{e}^{\pm at}t^{\nu_1+\nu_2+1/2} \cdot I_{\nu_1+\nu_2+1/2}(at)\,\mathrm{d}t < \int_0^x \mathrm{e}^{\pm at}t^{\nu_1}I_{\nu_1}(at)\,\mathrm{d}t.$$

By virtue of [34, p. 259, Eq. **10.43.7**] for $\Re(\nu) > -1/2$, we have

$$\int_0^x \mathrm{e}^{\pm t}t^\nu I_\nu(t)\,\mathrm{d}t = \frac{\mathrm{e}^{\pm x}x^{\nu+1}}{2\nu+1}\big(I_\nu(x)\mp I_{\nu+1}(x)\big),$$

and applying the substitution $at \mapsto u$ it turns out

$$\frac{\Gamma(\nu_1+3/2)}{\Gamma(\nu_1+\nu_2+2)}\left(\frac{(b^2-a^2)x}{2a}\right)^{\nu_2+1/2}\Big(I_{\nu_1+\nu_2+1/2}(ax) \mp I_{\nu_1+\nu_2+3/2}(ax)\Big) < I_{\nu_1}(ax)\mp I_{\nu_1+1}(ax).$$

Finally, inserting Horn's representation (5.1) into (5.2), we arrive at secondly stated inequality.

5.2 The use of the Turán inequality

It is the time to apply Turán inequality for the raw moments $m_s = \mathsf{E}\xi^s$, $s>0$ valid for non-negative random variables [25, p. 28, Eqs. (1.4.6)]

$$m_{s+r}^2 \le m_s\, m_{s+2r}, \qquad s, r > 0,$$

which is the consequence of the CBS inequality.

Define the Turánian ratio for the moment m_s with respect to some $r>0$ as

$$\mathscr{T}_r(m_s) = \frac{m_{s+r}^2}{m_s \cdot m_{s+2r}},$$

which transforms the last inequality into $\mathscr{T}_r(m_s) \le 1$. To get a bound for the Gaussian ${}_2F_1$, when we insert above

$$m_s = \frac{(b^2-a^2)^{\nu+1/2}\Gamma(2\nu+s+1)}{\Gamma(2\nu+1)\, b^{2\nu+s+1}}\, {}_2F_1[s], \tag{5.4}$$

where

$${}_2F_1[s] := {}_2F_1\left[\begin{matrix} \nu+\frac12(s+1),\ \nu+\frac{s}{2}+1 \\ \nu+1 \end{matrix} \Bigg| \frac{a^2}{b^2} \right].$$

This result holds for all $\Re(s) > -2\nu-1$.

Theorem 17 ([23, Proposition 2.1]). *For all $b>a>0$, $\nu>-1/2$ and $s, r>0$ we have*

$$\frac{\big\{{}_2F_1[s+r]\big\}^2}{{}_2F_1[s]\cdot {}_2F_1[s+2r]} \le \frac{\Gamma(2\nu+s+1)\Gamma(2\nu+s+2r+1)}{\Gamma^2(2\nu+s+r+1)}.$$

To derive another bound for ${}_2F_1[s]$ we take a moment inequality [15, p. 143, Theorem **192**]

$$\mathfrak{M}_r(h,p) < \mathfrak{M}_s(h,p), \qquad 0<r<s, \tag{5.5}$$

where

$$\mathfrak{M}_r(h,p) = \int_\alpha^\beta h^r(t)\, p(t)\, \mathrm{d}t,$$

for an integrable non-negative input function h, the integration interval (α,β) is either finite or infinite, and the non-negative weight p has integral $\int_\alpha^\beta p(t)\,\mathrm{d}t = 1$.

In our case the shorthand

$$\mathfrak{M}_s(x^s, f_I) = (m_s)^{1/s}$$

is adapted to the McKayI(a,b,ν) distribution, $(\alpha,\beta) = \mathbb{R}_+$. Inserting m_s from (5.4) into the integral moment inequality (5.5) we obtain

Theorem 18 ([23, Proposition 2.2]). *For all $b>a>0$, $\nu>-1/2$ and $s>r>0$ there holds true*

$$\frac{\big\{{}_2F_1[r]\big\}^{1/r}}{\big\{{}_2F_1[s]\big\}^{1/s}} \le \left(1-\frac{a^2}{b^2}\right)^{(\nu+1/2)(1/s-1/r)} \frac{(2\nu+1)_s^{1/s}}{(2\nu+1)_r^{1/r}},$$

where the hypergeometric terms are the same as above, that is

$${}_2F_1[s] := {}_2F_1\left[\begin{matrix} \nu+\frac12(s+1),\ \nu+\frac{s}{2}+1 \\ \nu+1 \end{matrix} \Bigg| \frac{a^2}{b^2} \right],$$

and $\min\big(\Re(s), \Re(r)\big) > -2\nu-1$.

5.3 Bounds for Horn Φ_3 when $\nu=0$

In this part of the presentation we establish a closed form representation for the cdf $F_I(x;a,b;0)$ in terms of the modified Bessel function of the first kind and the Horn confluent Φ_3 function of two variables. We also draw the reader's attention to the fact that the modified Bessel function of zeroth order I_0 is now involved in bounds.

Further, a two-sided bounding inequality is derived by the routine $F_I(x;a,b;0) \in [0,1]$.

Theorem 19 ([22, pp. 131–132, Proposition 3.2. and Corollary 3.3.]). *For all $x \ge 0$ and $b>a>0$ there holds*

$$F_I(x;a,b;0) = 1 + \mathrm{e}^{-bx}\Bigg[I_0(ax) - 2\Phi_3\left(1,1;\frac{x}{2}\big(b-\sqrt{b^2-a^2}\big); \frac{a^2x^2}{4}\right)\Bigg].$$

Moreover, we have

$$\frac12 I_0(ax) \le \Phi_3\left(1,1;\frac{x}{2}\big(b-\sqrt{b^2-a^2}\big); \frac{a^2x^2}{4}\right) \le \frac12\Big(\mathrm{e}^{bx} + I_0(ax)\Big).$$

6. F_I in terms of Srivastava–Daoust $\mathscr{S}$ function

Here we study the Srivastava–Daoust $\mathscr{S}$ function's monotonicity with the aid of the baseline function $H_1(x)$.

Theorem 20 ([21, p. 264, Theorem 3]). *For all $b>a>0$, $\nu>-\frac12$, we have the monotonicity property of the three variables Srivastava–Daoust function*

$$\mathscr{S}_\nu(x) = x^{2\nu+2}.$$

$$\mathscr{S}^{1:0;1;1}_{2:0;0;0}\left(\begin{matrix} [2\nu+2:1,2,1]:-;[\nu+\frac12:1];[1:1] \\ [2\nu+3:1,2,1],[2\nu+2:0,2,1]:-;-;- \end{matrix} \middle| \begin{matrix} -bx \\ (ax)^2 \\ bx \end{matrix} \right),$$

in the manner that for all $h>0,\ x\geq 0,$

$$\mathcal{S}_\nu(x) \leq \mathcal{S}_\nu(x+h) \leq \mathcal{S}_\nu(x) + \frac{2h\,(\nu+1)\Gamma(2\nu+1)}{(b^2-a^2)^{\nu+\frac12}}.$$

Proof. Consider the cdf $H_1(x)$ generated by the baseline cdf F_I when it is expressed *via* Exton X in (4.3) which is tracing back to [14, Theorem 4] (our Theorem 8).

Direct calculation, using the power series description of (4.3) and the Maclaurin series of the exponential term gives the following equality chain:

$$\begin{aligned}
H_1(x) &= \frac{(b^2-a^2)^{\nu+\frac12}}{h\,\Gamma(2\nu+1)} \sum_{j,m,n\geq 0} \frac{\left(\nu+\frac12\right)_m (1)_n (-b)^j}{(2\nu+2)_{2m+n}\, j!\, m!\, n!} \\
&\quad \cdot a^{2m} b^n \int_x^{x+h} t^{2\nu+1+j+2m+n}\,\mathrm{d}t \\
&= \frac{(b^2-a^2)^{\nu+\frac12}(x+h)^{2\nu+2}}{2h\,(\nu+1)\Gamma(2\nu+1)} \sum_{j,m,n\geq 0} \frac{(2\nu+2)_{j+2m+n}}{(2\nu+3)_{j+2m+n}} \\
&\quad \cdot \frac{\left(\nu+\frac12\right)_m (1)_n}{(2\nu+2)_{2m+n}} \frac{[-b(x+h)]^j}{j!} \\
&\quad \times \frac{[a(x+h)]^{2m}}{m!} \frac{[b(x+h)]^n}{n!} - \frac{(b^2-a^2)^{\nu+\frac12} x^{2\nu+2}}{2h\,(\nu+1)\Gamma(2\nu+1)} \\
&\quad \times \sum_{j,m,n\geq 0} \frac{(2\nu+2)_{j+2m+n}}{(2\nu+3)_{j+2m+n}} \\
&\quad \cdot \frac{\left(\nu+\frac12\right)_m (1)_n}{(2\nu+2)_{2m+n}} \frac{(-bx)^j}{j!} \frac{(ax)^{2m}}{m!} \frac{(bx)^n}{n!} \\
&= \frac{(b^2-a^2)^{\nu+\frac12}}{2h\,(\nu+1)\Gamma(2\nu+1)} \left\{\mathcal{S}_\nu(x+h) - \mathcal{S}_\nu(x)\right\}.
\end{aligned}$$

Now, the two-sided inequality follows by virtue of the aid of $0 \leq H_1(x) \leq 1;\ x\geq 0$.

7. F_I presented *via* Kampé de Fériet generalized hypergeometric function

7.1 Kampé de Fériet generalized hypergeometric function

The representation formula of the cdf F_I is now given in terms of the Kampé de Fériet function.

Theorem 21 ([19, p. 153, Corollary 3.]). *For all* $x \geq 0$, $\nu > -1/2$ *and* $b>a>0$ *we have*

$$\begin{aligned}
F_I(x;a,b;\nu) &= \frac{(b^2-a^2)^{\nu+1/2} x^{2\nu+1}}{\Gamma(2\nu+2)} \\
&\quad \cdot F^{0:1;1}_{1:0;0}\left[\begin{matrix} -:\nu+\frac12;\nu+\frac12 \\ 2\nu+2:-;- \end{matrix}\,\middle|\, x(a-b), -x(a+b)\right].
\end{aligned}$$

Recalling the reduction formula

$$\begin{aligned}
&\Phi_2\left(\nu+\frac12, \nu+\frac12; 2\nu+2; (a-b)x, -(a+b)x\right) \\
&\quad = F^{0:1;1}_{1:0;0}\left[\begin{matrix} -:\nu+\frac12;\nu+\frac12 \\ 2\nu+2:-;- \end{matrix}\,\middle|\, x(a-b), -x(a+b)\right],
\end{aligned}$$

we readily deduce the appropriate functional and uniform bounding inequalities for the KdF function, which hold for Horn confluent Φ_2 after Theorem 12.

8. Discussion. Final remarks

A. The CDF F_I for integer values of parameter ν can be presented in terms of the cdf of the non-central chi-square distribution $F_{n,\lambda}(x), x>0$, see [18, p. 2, Eq. (1.2)], (also see [8])

$$F_{n,\lambda}(x) = \mathrm{e}^{-\frac{\lambda}{2}} \sum_{k\geq 0} \frac{\lambda^k}{2^k\,\Gamma(\frac{n}{2}+k)k!}\,\gamma\left(\frac{n}{2}+k, \frac{x}{2}\right). \qquad (8.1)$$

The related result follows.

Theorem 22 ([14, pp. 6-7, Theorem 2]). *For all* $b>a>0$, $\nu\in\mathbb{N}_0$, *we have*

$$\begin{aligned}
&F_I(x;a,b;\nu) \\
&\quad = \frac{(-1)^\nu a^\nu}{2^\nu b^{2\nu+1}} \lim_{p\to 1} \mathfrak{D}_p^\nu \left\{\frac{1}{p}\mathrm{e}^{-\frac{a^2}{4b^2}\mathfrak{D}_p} F_{2\nu+2,-\frac{a^2}{4b^2}\mathfrak{D}_p}(2p\,b\,x)\right\},
\end{aligned}$$

where the exponential differential operator $\exp\left(-c\,\mathfrak{D}_p\right)$ *one defines by its associated formal Taylor series in which the differential operator*

$$h \mapsto \mathfrak{D}_p(h) = \frac{\partial}{\partial p}\frac{1}{p}(h)$$

is the building block and the CDF $F_{2\nu+2,-2c\mathfrak{D}_p}$ *is defined by* (8.1) *with* $c = a^2/(4b^2)$.

B. The article [14] contains further information regarding the rv $\xi \sim \mathrm{McKayI}(a,b,\nu)$. These results therein concern the unimodality, the Stieltjes moment problem regarding M-determinacy and the quantile function; consult the Section 5 in [14] for these probability and statistical topics.

C. In the abstract it was mentioned the existence of the familiar inter-connection formula between the modified Bessel function of the first kind and Kummer's confluent hypergeometric function, which reads [34, p. 255, Eq. 10.39.9.]

$$I_\nu(z) = \frac{z^\nu}{2^\nu\,\Gamma(\nu+1)}\,{}_0F_1\left(-;\nu+1;\frac{z^2}{4}\right).$$

Since $I_\nu(\mathrm{i}z) = \mathrm{i}^\nu J_\nu(z)$, we have

$$J_\nu(z) = \frac{z^\nu}{2^\nu\,\Gamma(\nu+1)}\,{}_0F_1\left(-;\nu+1;-\frac{z^2}{4}\right).$$

The list of bounding inequalities for the Bessel function of the first kind is long, see e.g. [3, pp. 11–13, Section 1.7] and the referenced publications; also consult [2]. Moreover, we recall here an upper bound which is not contained in that monograph. Namely, denote by $j_{\nu,1}$, the smallest positive zero of the Bessel function of the first kind of the order ν, and consider the open Cassinian oval

$$\mathfrak{C}_{\nu,\lambda} = \left\{z\in\mathbb{C} : \left|z^2 - j_{\nu,1}^2\right| < j_{\nu,1}\,\frac{1-\lambda}{1+\lambda}\right\}, \quad \lambda\in(0,1).$$

Then, we have [36, p. 194, Theorem 1.]

$$|J_\nu(z)| \le \frac{|z|^\nu}{2^\nu\,\Gamma(\nu+1)}\,\mathrm{e}^{-\frac{\lambda|z|^2}{4(\nu+1)}}, \quad z\in\mathfrak{C}_{\nu,\lambda}.$$

Conjunction of above linked and the presented bounds imply novel bounding inequalities for the Kummer confluent hypergeometric function.

D. The Gaussian hypergeometric function of the special form

$${}_2F_1\left[\begin{matrix} a,\, b \\ a+b\end{matrix}\,\middle|\, z\right]$$

is call zero-balanced. There are famous asymptotic expansion results for $z\to 1$ which belong to Gauss himself, then to Ramanujan, for instance. We draw the reader's attention to the article by Ponnusamy and Vuorinen [40], in which the authors extend and refined Ramanujan's results for ${}_2F_1$ when $a,b,c > 0$ and $c < a+b$. Their monotonicity results are actually bounding inequalities for the Gaussian hypergeometric function, among others.

E. The most cited article devoted to the inequalities [26] for the hypergeometric functions belongs to Yadell L. Luke. Luke gave several new results using Padé–, or rational approximation technique not only for the Gaussian ${}_2F_1$, but also for the generalized hypergeometric ${}_{q+1}F_q$, incomplete Gamma function, confluent hypergeometric ${}_pF_p$ functions, particular Meijer G functions, complete elliptic integrals of the first and second kind $\mathbf{K}(k), \mathbf{E}(k)$. Bilateral bounding inequality for the modified Bessel function of the first kind he inferred *via* the formula

$$I_\nu(z) = \frac{z^\nu\,\mathrm{e}^z}{2^\nu\,\Gamma(\nu+1)}\,{}_1F_1\left[\begin{matrix}\nu+\frac12 \\ 2\nu+1\end{matrix}\,\middle|\,-2z\right],$$

which results in the relation [26, p. 63, Eq. (6.23)] for all $z > 0$ and for all $\nu \ge -\frac12$. He also derived bounds for the modified Bessel function $K_\nu(z)$, and for the parabolic cylinder function $D_\nu(z)$. It is also worth to mention Carlson's article [9], Section 3. is devoted to inequalities for ${}_1F_1, {}_2F_1$ and ${}_2F_0$.

Some Luke's results are extended by Pogány and Tomovski in [39] to generalized hypergeometric functions in the case $p \le q$. Finally, Pogány and Srivastava published an exponential two-sided inequality for the Fox–Wright functions ${}_p\Psi_q$ in [38, p. 133, Theorem 4.].

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Does This Really Make Sense?

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Abstract. A simple, apparent paradox demands clear probabilistic analysis to unravel.

Keywords. Probability, Expected value, Sample space.

Subject Classification 2020: 60-02, 60-08.

A simple, apparent paradox can be just the thing to generate student interest. Consider this: Face up on the table before you is a card showing a positive integer. You are told that each card in the deck from which it was drawn bears a positive integer on one side and double that integer on the other. You can either accept the displayed amount in dollars, or turn the card over and receive the amount on the other side. Which will you choose?

At first glance it would seem to make no difference. Whoever dealt the card might just as well have turned it over while placing it down. But a bit of calculation contradicts this. If X is the amount shown, then the reverse side will be either $2X$ or $0.5X$. The expected value of turning the card over is therefore $0.5(2X)+0.5(0.5X) = 1.25X$. Evidently you should turn it over!

Wait a minute—does this make any sense? What's going on here? [1,2,3] Students of course should be encouraged to investigate this apparent paradox, but here is an analysis:

Unfounded Assumptions The most obvious unjustified assumption here is that the cards—label them $[X/2X]$ and $[0.5X/X]$—are equally likely. This, however, is just a prelude to the murkiness of the above narrative. We don't even know if both cards are included in the deck, although at least one of them must be. And what about the deck itself? Is it finite or infinite? Indeed, could it actually *be* infinite? What precise cards are contained in it, and what is their probability distribution? Finally, how much of this information is known to you, the "player"?

Let's try to put this in some order, first clearly defining the deck as a sample space.

Finite or infinite? There is nothing to prevent the deck's having a countably infinite number of cards, at least in theory. Label the cards with the positive integers and assign the probability 2^{-n} to card n (i.e., $P(n) = 2^{-n}$). By summing the series we see that the full deck will be a sample space with probability 1. Indeed, any geometric series with $k > 1$, suitably normalized such that $P(n) = (k-1)(k^{-n})$, will produce a valid sample space, as will the assignment $P(n) = 1/n(n+1)$, or any series with a finite sum, appropriately normalized.

In practice, however, it's hard to imagine how an individual card could be selected from an infinite deck. Clearly the deck could not be physical. And while software could be written to choose one card from a virtual deck of any size, that size would have to be finite or the program would never stop. Thus an infinite deck is not practically feasible, and we will limit our discussion to finite decks.

The player's knowledge Before investigating the deck's composition, let's dismiss the question of how much the player knows. We are interested in well defined probabilities and expected values, which can be calculated only if the player knows everything about the deck except the number which is face down. If, for example, the player were ignorant of whether the deck contained a specific, relevant card, then no calculation would be possible. We therefore assume the player has enough information to make probabilistic decisions in an attempt to maximize earnings.

Example of a simple deck Regardless of our skepticism, the strategy of turning the card over—at least *almost* always—is genuinely advantageous and presents no paradox. Start by considering a deck consisting of four cards: [1,2], [2,4], [4,8], [8,16]. Call this a "doubling string of length 4". Assume the card is drawn at random, so each of the eight sides is likely to be face up with probability 1/8. If the player does *not* turn the card over, the expected value, or the average earning if the game is played many times, is

$$E\text{ (no turn)} = (1/8)(1+2+2+4+4+8+8+16) = 45/8 = 5.625.$$

If the card is always turned,

$$E\text{ (turn)} = (1/8)(2+1+4+2+8+4+16+8) = 45/8 = 5.625.$$

Thus there is no advantage in always turning. But note the final 8 in the parenthesis for E (turn). This results from a face up 16, which, if the player adheres strictly to the requirement to turn, becomes an 8. E (turn) is equal to E (no turn), and any sense of paradox has disappeared. But where is the turning advantage we saw above, the $1.25X$? It's still there in the first three turns, but gets completely cancelled by the disadvantageous turn from 16 to 8. To avoid this the player has to violate the rule of turning whenever the card [8,16] shows the 16, for the player knows there is no card [16,32], and thus to maximize earnings has to stay with the 16. This increases E (turn) by the amount (1/8)(8), to 6.625, a value which we can call E (turn+).

So, again, there *is* a distinct immediate advantage in turning when any number but the last in a string is face up, but in the long run this advantage is cancelled by the disadvantageous turn when the last number is face up. That is, E (turn) = E (no turn), but when maximizing earnings,

$$E(\text{turn+}) > E\ (\text{no turn}).$$

Generalization An alert student might discover that the cards needn't be precisely $[X, 2X]$ in order for the above reasoning to apply. Let the cards be $[A_1, A_2]$, $[A_2, A_3]$ .... $[A_{n-1}, A_n]$, with all A_k's positive, $A_k < A_{k+1}$ and $(A_k - A_{k-1}) < (A_{k+1} - A_k)$ whenever defined. This last inequality assures that if the player is in front of a card reading A_k the expected value will be greater than A_k if the card is turned. Call this a "turnable string".

For a turnable string of length n, E (no turn) and E (turn) both equal $(1/2n)(A_1{+}2A_2{+}2A_3{+}\cdots{+}2A_{n-1}{+}A_n)$. But again, if the player violates the turn rule on the last card—knowing that there is no card $[A_n, A_{n+1}]$—then E (turn) increases by $(1/2n)(A_n - A_{n-1})$. The doubling string is a particular example of this.

Broader generalizations So far the deck has been limited to one string. A deck composed of several turnable strings with no card included in more than one string can be analyzed in the same way as a deck composed of a single turnable string. If the turn rule is strictly followed, E (no turn) will be equal to E (turn) and if the player does *not* turn the last card on any string, the advantage will be added to this expected value to give E (turn+) > E (no turn).

If a card is duplicated within a string, it can be readily shown that E (turn) and E (no turn) remain equal. However, the player, who tries to maximize earnings, will now change strategy when faced with a number that could be on a duplicate card. In considering whether to turn that card over, there is no longer a 0.5 probability of increase or decrease. If there are two of the duplicate card, the probabilities change to 2/3 and 1/3. Indeed, if there are m_L copies of the left (lower) repeated card and m_R copies of the right repeated card, then the probabilities for estimating whether to turn or not are $m_L/(m_L + m_R)$ and $m_R/(m_L + m_R)$. As long as the player knows the exact composition of the deck, optimum calculations can be performed for any particular situation. And while the benefit of turning or not turning may vary depending upon what number is face up, as long as the last (highest) number in each string is turned—and this of course will not serve to maximize earnings—the expected value of always turning or never turning will be equal.

The above description is a complete account of any deck composed of turnable strings. There are also decks in which two strings contain cards that match in only one of the numbers, and while this puts more computational demand on the player, the expected values of E (no turn) and E (turn) remain equal.

Paradox or not? In summary, we see that given complete knowledge of the deck, the player can in fact gain an advantage by turning over any card other than the last card in a string. If played many times, the sum of all these advantages is nullified if the player also turns the last card whenever the highest number appears. In this case E (turn) = E (no turn). The player's entire advantage lies in recognizing the singularity of the last card, and violating the turn rule so that E (turn+) > E (no turn).

Summary Investigation of a simple game promotes student understanding of some basic concepts in probability: sample space, well-defined probabilities, and expected value.

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Various Representation Dimensions associated with a Finite Group

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Abstract. To a finite group G, one can associate several notions of dimension (or degree). In this survey, we attempt to bring together some of the notions of dimension or degree defined using representations of the group in General Linear Groups and permutation groups. These are embedding degree, minimal faithful irreducible character degree, minimal faithful permutation representation degree, minimal faithful quasi-permutation representation degree and essential dimension. We briefly present the progress in understanding these notions and the related problems.

Keywords. Embedding degree, minimal faithful irreducible character degree, minimal faithful permutation representation degree, minimal faithful quasi-permutation representation degree, essential dimension.

Subject Classification 2020: 20C15.

1. Introduction

Let G be a finite group. As soon as we are introduced to a group via axiomatic definition in our undergraduate courses, several questions come to mind. One among them is can we think of a finite group as a subgroup of some concrete group? Indeed an answer is provided by Cayley's theorem that a finite group G of size n can be embedded inside S_n using the left multiplication map. We can also use a similar method to embed G inside $GL_n(\mathbb{C})$ giving rise to the regular representation. The underlying concept is that we supposedly understand the structure of symmetric groups and linear groups better which involve combinatorics and linear algebra, respectively. One can further ask questions the other way around that for a fixed d and field k which finite groups occur as subgroups of $GL_d(k)$ and S_d. These questions date back to Minkowaski (1887) and Jordan (1878) whose contribution is beautifully discussed by Serre in his book Finite Groups: an Introduction [S22].

Indeed, these ideas led to the definition of several different kinds of "dimensions" or "degrees" associated with a group G. In this note, we briefly explore what is embedding degree $\delta(G)$, minimal faithful irreducible character degree $\delta_{irr}(G)$, minimal faithful permutation representation degree $\mu(G)$, minimal faithful quasi-permutation representation degree $q(G)$, and essential dimension $ed_K(G)$. These notions are studied in literature some of which we aim to bring together. We explain them through some easy examples and provide updates on related problems. Throughout this article, p denotes a prime, and $\mathbb{C}$ is the underlying field unless stated otherwise.

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2. Embedding Degree

The *embedding degree*, or the representation dimension, of a finite group G, denoted $\delta(G)$ or by rdim(G), is the minimal dimension of a faithful complex representation of G.

Alternatively, $\delta(G)$ is the smallest positive integer n such that G embeds into $GL_n(\mathbb{C})$, i.e., G is isomorphic to a linear group of degree n. The determination of the embedding degree of a group is one of the classical and challenging problems in the representation theory of finite groups, and its study has found many applications, for example, its connections with the *essential dimension*, $\mathrm{ed}(G)$, of a group G. We discuss the essential dimension of G in Section 6. Note that the determination of all finite groups with representation dimension δ is equivalent to the determination of all the finite subgroups of $GL_\delta(\mathbb{C})$ up to isomorphism.

Motivated by the relation between the embedding degree and the essential dimension, in 2011, Cernele et al. [CKR11] initiated the study of the embedding degree and obtained an upper bound for $\delta(G)$ of a p-group G, where p is a prime. They proved that if the order of G is p^n and the rank of $\mathcal{Z}(G)$ is r, then for almost all pairs (p, n), the maximum value of $\delta(G)$ is $f_p(n)$, where

$$f_p(n) = \max_{r\in\mathbb{N}} \left(r p^{\lfloor (n-r)/2 \rfloor} \right).$$

The exceptional cases are of (p, n) being $(2, 5)$, $(2, 7)$ and $(p, 4)$ where p is odd. In these cases the maximum value of $\delta(G)$ is 5, 10, and $p + 1$, respectively.

Example 2.1. *If G is a cyclic group then $\delta(G) = 1$. In fact, $\delta(G) = 1$ if and only if G is cyclic. However, if $G \cong C_2 \times C_2$, then $\delta(G) = 2$.*

Example 2.2. *When G is dihedral group D_n, Q_8, and $C_n \times C_m$ where n is not coprime to m then $\delta(G) = 2$. Note that $\delta(G) = 2$ amounts to finding all finite subgroups of $GL_2(\mathbb{C})$.*

Utilizing a result of [MR10], Bardestani et al. [BMS16] proved that if G is a p-group and ρ is a faithful representation of G of least dimension, then ρ decomposes into exactly r irreducible representations ρ_i $(1 \le i \le r)$ of G, where r is the rank of $\mathcal{Z}(G)$. Then

$$\begin{aligned} 1 &= \ker(\rho) \\ &= \bigcap_{i=1}^{r} \ker(\rho_i) \text{ and } \bigcap_{i=1, i\neq j}^{r} \ker(\rho_i) \neq 1 \\ &\qquad \text{for each } j \ (1 \le j \le r). \end{aligned}$$

Since there exists a faithful character of the least degree in G, $\delta(G)$ may be restated as

$$\delta(G) = \min\left\{ \sum_{\chi\in X} \chi(1) : X \subset \mathrm{Irr}(G) \text{ with } |X| = r \text{ such that } \bigcap_{\chi\in X} \ker(\chi) = 1 \text{ and } \bigcap_{\chi\in Y} \ker(\chi) \neq 1 \text{ for all } Y \subsetneq X \right\}.$$

Bardestani et al. [BMS16] further computed the embedding degree of the Heisenberg group over $\mathcal{O}/\mathcal{I}^n$, where $\mathcal{O}$ is the associated ring of integers of non-Archimedean local field, and $\mathcal{I}$ is the maximal ideal of $\mathcal{O}$. Moretó [M21] proved that for a finite group G, $\delta(G) \le \sqrt{|G|}$ or $\frac{3}{\sqrt{8}}\sqrt{|G|}$. Further, Moretó classified groups for which $\delta(G) = \sqrt{|G|}$; these groups turn out to be 2-groups. It also proved that the embedding degree of a finite abelian group is the rank of the group. This implies that the embedding degree for finite groups may be arbitrarily large. In [KKS23], Kaur et al. computed the embedding degree for various classes of groups, such as some finite p-groups (p is a prime), certain direct products of finite groups, odd order groups with exactly two nonlinear irreducible characters of each degree, and finite groups whose all nonlinear irreducible characters are of distinct degree. They also provide GAP [GAP] code for the computation of the embedding degree. Lübeck [L01] and Tiep and Zalesskii [TZ96] have studied representation dimension for finite groups of Lie type.

3. Minimal Faithful Irreducible Character Degree

For a group G, the *minimal faithful irreducible character degree* of G, $\delta_{\mathrm{irr}}(G)$, is defined as

$$\delta_{\mathrm{irr}}(G) := \min\{\dim\rho : \rho \text{ is an irreducible faithful representation of } G\}.$$

Note that a group may not have a faithful irreducible character and hence $\delta_{\mathrm{irr}}(G)$ may not exist.

Example 3.1. *For $G = C_2 \times C_2$, $\delta_{irr}(G)$ does not exist.*

The study of the faithful irreducible representations of a group has been an intriguing area of research. Since the beginning of 20th century, many researchers have determined various characterizations for groups which possess a faithful irreducible character; Szechtman provides a detailed history of the problem in [S16]. Kitture and Pradhan [KP22] obtained the degrees of faithful irreducible representations for a class of metabelian groups. When a faithful irreducible character of G exists (say χ), we have the inequality $\delta(G) \leq \delta_{\text{irr}}(G)$. However, the quantity $\delta(G)$ may be strictly less than $\delta_{\text{irr}}(G)$.

Example 3.2. *We have* $\delta(A_4 \times D_{10}) = 5$ *and* $\delta_{\text{irr}}(A_4 \times D_{10}) = 6$ *(see Example 2.4 of [KKS23]).*

In the literature, the minimal faithful irreducible character degree for various classes of groups has been studied. In the case of non-abelian finite simple groups, every nontrivial character is faithful. Hence, for these groups, the embedding degree is equal to the minimal degree of a nonlinear irreducible character. By Lemma 3.1 of [BTVZ17], we get $\delta(A_n) = \delta_{\text{irr}}(A_n) = n - 1$ for $n \geq 15$. It is easy to compute from GAP that $\delta(A_n) = n - 1$ for $6 \leq n \leq 14$, and $\delta(A_5) = 3$. Further, from a result of Rasala [R77], we get $\delta_{\text{irr}}(S_n) = n - 1$ for $n \geq 5$. Since the standard character of S_n is faithful and has degree $n - 1$, it follows that $\delta(S_n) = n - 1$ for $n \geq 5$. From Steinberg's work in [S51], we get that $\delta_{\text{irr}}(GL(2, q)) = q - 1$, $\delta_{\text{irr}}(GL(3, q)) = q^2 + q$ and $\delta_{\text{irr}}(GL(4, q)) = (q + 1)(q^2 + 1)$. A group G is called normally monomial if every irreducible character of G is induced from a linear character of some normal subgroup of G. In [M99], Mann proved that all faithful irreducible characters of a normally monomial group are of the same degree, which is the maximum degree of all irreducible characters of the group. As a result, if G is a normally monomial p-group with the cyclic center, then $\delta_{\text{irr}}(G) = \max \text{cd}(G)$, where $\text{cd}(G)$ denotes the set of character degrees of G. Kaur et al. [KKS23] proved that if G is a finite nilpotent group with the cyclic centre, then $\delta(G) = \delta_{\text{irr}}(G)$. Hence, we also obtain the embedding degree for any normally monomial p-group with a cyclic centre from Mann's result. In [KKS23], Kaur et al. obtained a condition when $\delta(G_1 \times G_2) = \delta_{\text{irr}}(G_1 \times G_2) = \delta_{\text{irr}}(G_1)\delta_{\text{irr}}(G_2)$, where G_1 and G_2 are finite non-abelian groups.

Question 1. *Characterize groups* G_1 *and* G_2 *such that* $\delta(G_1 \times G_2) = \delta_{\text{irr}}(G_1 \times G_2)$.

Kaur et al. also computed $\delta(G)$ and $\delta_{\text{irr}}(G)$ where G is a finite non-abelian group of odd order with exactly two nonlinear irreducible characters of each degree or a finite non-abelian group whose nonlinear irreducible characters have distinct degrees.

4. Minimal Faithful Permutation Representation Degree

Cayley's theorem states that every flite group is isomorphic to a subgroup of a symmetric group. It is interesting to find the least positive integer n such that G is embedded in the symmetric group S_n. Such n is called the *minimal faithful permutation representation degree* of G, denoted by $\mu(G)$. One can easily observe that for a group G, we have $\delta(G) \leq \mu(G)$. It is well known that the minimal faithful permutation representation degree of G is given by

$$\mu(G) = \min\left\{ \sum_{i=1}^{k} |G : G_i| : G_i \leq G \text{ for } 1 \leq i \leq k \text{ and } \bigcap_{i=1}^{k} \text{Core}_G(G_i) = 1 \right\},$$

where $\text{Core}_G(G_i)$ is the core of G_i in G, defined as the largest normal subgroup of G contained in G_i. Here $\mathcal{H} = \{G_1, \ldots, G_k\}$ is called a *minimal faithful permutation representation* of G. Let H be a group and $\{H_1, H_2, \ldots, H_s\}$ be a minimal faithful permutation representation of H. Then one can show that

$$\{G_1 \times H, \ldots, G_r \times H, G \times H_1, \ldots, G \times H_s\}$$

is a minimal faithful permutation representation of $G \times H$.

Example 4.1. *One can show the following:* $\mu(C_p) = p$, *p a prime;* $\mu(C_3 \times C_3) = 6 = \mu(C_3) + \mu(C_3)$; $\mu(C_6) = 5$; $\mu(D_4) = 4$ *(*$|D_4| = 8$*);* $\mu(Q_8) = 8$.

Karpilovsky [K70] first computed $\mu(G)$ for a finite abelian group G. Johnson [J71] proved that $\mu(G) = |G|$ if and only if G is isomorphic to a cyclic p-group (for a prime p), a generalized quaternion group or the non-cyclic group of order 4. Johnson also computed the cardinality of a minimal faithful permutation representation of a p-group. Various researchers investigated the relation between $\mu(G \times H)$ and $\mu(G) + \mu(H)$ for two groups G and H. Johnson [J71, Proposition 2] proved that for any two groups G and H,

$$\mu(G \times H) \leq \mu(G) + \mu(H),$$

and the reverse inequality holds whenever G and H have coprime orders. Wright [W75] improved this result and proved that for all non-trivial $G, H \in$ {nilpotent, symmetric, alternating or dihedral groups}, $\mu(G \times H) = \mu(G) + \mu(H)$. Wright further provided an example of groups G and H such that $\mu(G \times H) \neq \mu(G) + \mu(H)$ (see [W75, Section 5]). Easdown and Praeger [EP88] proved that for finite simple groups G and H, $\mu(G \times H) = \mu(G) + \mu(H)$. Saunders [S09] produced an infinite family of examples of permutation groups G and H where $\mu(G \times H) < \mu(G) + \mu(H)$. In [S10], Saunders computed the minimal faithful permutation degree of a class of finite reflection groups and proved that they form examples where the minimal degree of a direct product is strictly less than the sum of the minimal degrees of the direct factors. Easdown and Saunders [ES16] further proved that 10 is minimal in the sense that $\mu(G \times H) = \mu(G) + \mu(H)$ for all groups G and H such that $\mu(G \times H) \leq 9$. Easdown and Hendriksen [EH16] proved that if G and H are finite groups, then $\mu(G \rtimes H) \leq |G| + \mu(H)$.

Several researchers, for example, Babai et al. [BGP93] and Franchi [F11] investigated bounds of $\mu(G)$ for a group G. Babai et al. [BGP93] proved that $\nu(G) \geq |G|/\mu(G) \geq f(\nu(G))$, where $\nu(G)$ is the index of the largest cyclic subgroup of prime-power order in G and $f(n) = e^{(c\sqrt{\log n})}$ for a constant $c > 0$. Franchi [F11] proved that if G is a finite p-group with an abelian maximal subgroup, then $\mu(G/G') \leq \mu(G)$. Berkovich [YB99] has obtained various results by investigating the relation between $\mu(G)$ and $i(G)$ for a group G, where $i(G) = \min\{|G : H| : H < G\}$ if $G > 1$ and $i(G) = 1$ if $G = 1$. Hall and Senior [HS64] listed minimal degree faithful representations for the groups of order dividing 2^6. Lemieux [LThesis] determined the values of $\mu(G)$ for the groups of order dividing p^4.

For a group G and its normal subgroup N, various researchers have investigated the relation between $\mu(G)$ and $\mu(G/N)$. In [N86], Neumann took G to be the direct product of $k > 1$ copies of the dihedral group of order 8 to show that $\mu(G/N) > \mu(G)$ for some normal subgroup N of G. Easdown and Praeger [EP88] produced further examples of such groups G with quotient groups G/N such that $\mu(G/N) > \mu(G)$. A group G is called exceptional if there exists a normal subgroup N such that $\mu(G/N) > \mu(G)$; here N is called a distinguished subgroup and G/N is called a distinguished quotient. Kovács and Praeger proved that $\mu(G/N) \leq \mu(G)$ if G/N has no nontrivial abelian normal subgroup. Holt and Walton [HW02] proved that there exists a constant c such that $\mu(G/N) \leq c^{\mu(G)-1}$ for all finite groups G and all normal subgroups N of G. In [LThesis, SL07], Lemieux proved that there are no exceptional p-groups of order less than p^5, and also produced an exceptional group of order p^5 where p is an odd prime. Chamberlain [C18] independently proved that there are no exceptional p-groups of order less than p^5. In [S14], Saunders computed the minimal faithful permutation degree of the irreducible Coxeter groups and exhibited examples of new exceptional groups. Britnell et al. [BSS17] classified all exceptional p-groups of order p^5 where p is a prime and proved that the proportion of groups of order p^5 that are exceptional is asymptotically 1/2. Due to this observation, they emphasize that the established term exceptional for these groups may be less appropriate. However, they have not commented on the case of groups of order p^6.

5. Minimal Faithful Quasi-permutation Representation Degree

A quasi-permutation matrix is defined to be a complex square matrix with a non-negative integral trace. In 1963, Wong [W63] defined a Quasi-permutation group to be a finite group in which every element is a quasi-permutation matrix. The above terminology allows us to have two more degrees.

Definition 5.1. *The minimal degree of a faithful representation of G by quasi-permutation matrices over the field $\mathbb{Q}$ is denoted by $q(G)$. The minimal degree of a faithful representation of G by complex quasi-permutation matrices is denoted by $c(G)$.*

Since every permutation matrix is a quasi-permutation matrix, it is easy to see that $c(G) \leq q(G) \leq \mu(G)$, i.e., $c(G)$ and $q(G)$ provide a lower bound for $\mu(G)$. The above inequalities are sometimes strict, and sometimes not.

Example 5.2. *Take $G_1 = SL(2,5)$, $G_2 = C_6$, $G_3 = Q_8$ (the quaternion group of order 8), and $G_4 = D_4$ (the dihedral group of order 8). Then $c(G_1) = 8$, $q(G_1) = 16$ and $\mu(G_1) = 24$ (see [BGHS94]), $c(G_2) = q(C_2) = 4$ and $c(G_2) = 5$,*

$c(G_3) = 4$ *and* $q(G_3) = \mu(G_3) = 8$, *and* $c(G_4) = q(G_4) = \mu(G_4) = 4$. *Of course, we have* $\delta(G) \leq c(G)$.

Burns et al. [BGHS94] initiated the computation of $c(G)$ and $q(G)$ for abelian groups and proved that if G is a finite abelian group, then $c(G) = q(G) = \mu(G)$ if and only if G has no direct factor of order 6. Behravesh [B97b] improved the above result and proved that if $G \cong \prod_{i=1}^{k} C_{p^{a_i}}$ and n is maximal such that G has a direct factor isomorphic to C_6^n, then $c(G) = q(G) = \mu(G) - n$. Behravesh [B97a] provided an algorithm for computing $c(G)$ and $q(G)$ for a finite group G. Behravesh and Ghaffarzadeh [BG11] improved the said algorithm and proved the following:

Lemma 5.3 ([BG11, Lemma 2.2]). *Let G be a finite group. Let $X \subset \mathrm{Irr}(G)$ be such that $\bigcap_{\chi\in X} \ker(\chi) = 1$ and $\bigcap_{\chi\in Y} \ker(\chi) \neq 1$ for every proper subset Y of X. Define $\xi_X = \sum_{\chi\in X}\sum_{\sigma\in\Gamma(\chi)} \chi^\sigma$ and $\xi'_X = \sum_{\chi\in X}\left[m_{\mathbb{Q}}(\chi)\sum_{\sigma\in\Gamma(\chi)}\chi^\sigma\right]$, where for $\chi \in \mathrm{Irr}(G)$, $\Gamma(\chi) = \mathrm{Gal}(\mathbb{Q}(\chi) : \mathbb{Q})$ and $m_{\mathbb{Q}}(\chi)$ is the Schur index of χ over $\mathbb{Q}$. Let $m(\xi_X)$ and $m(\xi'_X)$ be the absolute value of the minimum value that ξ_X and ξ'_X take over G respectively. Then*

$$c(G) = \min\{\xi_X(1) + m(\xi_X) \mid X \subset \mathrm{Irr}(G) \text{ satisfying the above property}\}, \text{ and}$$

$$q(G) = \min\{\xi'_X(1) + m(\xi'_X) \mid X \subset \mathrm{Irr}(G) \text{ satisfying the above property}\}.$$

The proof of Lemma 5.3 contains a minor error, and Prajapati and Udeep have included a correct proof in [PU23a]. Behravesh and Ghaffarzadeh [BG11] proved that if G is a p-group (p is a prime), then $q(G) = \mu(G)$. Moreover, if $p \neq 2$, then $c(G) = q(G) = \mu(G)$. Hence if $G = H \times K$, where H and K are p-groups (p an odd prime), then $c(G) = c(H) + c(K)$ and $q(G) = q(H) + q(K)$. Ghaffarzadeh and Abbaspour [GA12] further proved that the above result is true even for 2-groups H and K. In the same article, they proved that if $X \subset \mathrm{Irr}(G)$ satisfying the hypothesis of Lemma 5.3 such that $c(G) = \xi_X(1) + m(\xi_X)$, then the rank of $\mathcal{Z}(G)$ is the cardinality of X.

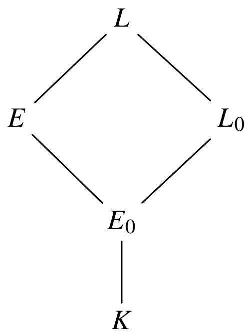
Various researchers have worked out $c(G)$ and $q(G)$ for different classes of groups. In [B97a], Behravesh proved that if G is a p-group of nilpotency class 2 with cyclic center, then $c(G) = |G/\mathcal{Z}(G)|^{1/2}|\mathcal{Z}(G)|$. Behravesh computed $c(G)$, $q(G)$ and $\mu(G)$ when G is $SL(2,q)$ or $PSL(2,q)$ in [B99]; Darafsheh et al. [DGDB01] computed the degrees for $G = GL(2,q)$, and Darafsheh and Ghorbany [DG03] computed the degrees when G is $SU(3,q^2)$ or $PSU(3,q^2)$.

One can see significant progress in the case of p-groups, where p is a prime. Since for p-groups G of odd order, $\mu(G) = c(G) = q(G)$, the quantities $c(G)$ and $q(G)$ for groups G of order p^4 (p odd) are derived from Lemieux's work [LThesis]. In [B00], Behravesh computed $c(G)$ when G is a metacyclic p-group with a non-cyclic center. Behravesh and Delfani [BD18] claim to list $c(G)$ for the groups of order p^5 (p odd); however, their work has many inaccuracies. Prajapati and Udeep have computed $\mu(G)$ and $c(G)$ for groups G of odd order p^5 in [PU24]. Pradhan and Sury [PS22] studied the quasi-permutation representations as well as the degrees $c(G)$, $q(G)$ and $\mu(G)$ for the holomorph of cyclic p-groups, where p is a prime. In [PU23a,PU23b], Prajapati and Udeep studied $c(G)$, $q(G)$ and $\mu(G)$ for various classes of non-abelian p-groups such as VZ p-groups, Camina p-groups, and the groups with cyclic center. In [PU23b], authors prove that if G is either a VZ p-group with $d(G') = d(\mathcal{Z}(G))$, or a Camina p-group, then $c(G) = |G/\mathcal{Z}(G)|^{1/2}c(\mathcal{Z}(G))$. In [O'BPU24], O'Brien et al. computed $c(G)$ and $\mu(G)$ for all the groups of order p^6 (p odd); they also provide a minimal faithful permutation representation for each group. In [PU23c], Prajapati and Udeep establish equality between $c(G)$ and a $\mathbb{Q}_{\geq 0}$-sum of codegrees of some irreducible characters of a non-abelian p-group G of odd order. Note that for $\chi \in \mathrm{Irr}(G)$, codegree of χ is defined as $\frac{|G/\ker(\chi)|}{\chi(1)}$. In the same article, Prajapati and Udeep prove that if G is a GVZ p-group with a cyclic center, then $c(G) = |G/\mathcal{Z}(G)|^{1/2}|\mathcal{Z}(G)|$.

Question 2. *Characterize p-groups G for which $c(G) = |G/\mathcal{Z}(G)|^{1/2}|\mathcal{Z}(G)|$.*

6. Essential Dimension

Buhler and Reichstein [BR97] established the notion of essential dimension in the following way. Let K be a field and L/L_0 be a finite separable extension containing K. We say L/L_0 is defined over E_0 if there is a field E such that $K \subset E_0 \subset E \subset L$, $[E : E_0] = [L : L_0]$ and $L = EL_0$.



The essential dimension of L/L_0 is given by

$$ed_K(L/L_0) = \min tr.deg_K E_0$$

where E_0 runs through all the intermediate fields over which L/L_0 is defined, and $tr.deg_K$ denotes the transcendence degree of E_0 over K. Further, Buhler and Reichstein defined the essential dimension of a finite group, denoted $ed_K(G)$, as the essential dimension of any Noether extension $K(V)/K(V)^G$, where $G \to GL(V)$ is a faithful representation of G and $K(V)$ is the function field of the affine space V over K. The integer $ed_K(G)$ is the smallest number of algebraically independent parameters required to define Galois G-algebras over any field extension of K. There is another functorial definition of essential dimension due to Merkurjev [M00] in his unpublished notes, which Berhuy and Favi later mentioned in their paper (see [BF13]) too.

Various researchers have studied essential dimensions in the recent past. Buhler and Reichstein [BR97] proved that $ed_K(H_1 \times H_2) \leq ed_K(H_1) + ed_K(H_2)$ for groups H_1 and H_2. They also proved that if K is a field containing all roots of unity, then $ed_K(G) = 1$ if and only if G is isomorphic to a cyclic group or a dihedral group D_{2m} where m is odd. Ledet [L07] further proved that a finite group G has essential dimension 1 over an infinite field K if and only if there exists an embedding of G in $GL_2(K)$ such that the image of G contains no scalar matrices other than the identity. In [R00], Reichstein computed essential dimensions for various classes of groups such as Symmetric groups, Orthogonal groups, General linear groups, etc. In [BF13], Berhuy and Favi proved that for a finite group G, $ed_K(G) \leq \delta_K(G)$, where $\delta_K(G)$ is the embedding degree of G over K. In fact, Karpenko and Merkurjev [KM08] proved that if G is a p-group and K is a field of characteristic different from p containing a primitive p^{th} root of unity, then $ed_K(G) = \delta_K(G)$. Florence [F08] computed the essential dimension of a cyclic p-group where K in a field of characteristic not p, containing the p^{th} roots of unity. Further, Wong [W11] computed an upper bound of the essential dimension of the cyclic group of order $p_1^{n_1} \cdots p_r^{n_r}$ over a field K of characteristic different from p_i containing all the primitive p_i^{th} roots of unity, where p_i's are distinct prime numbers. Brosnan et al. [BRV10] studied the essential dimension for the spinor group. Readers can see detailed survey articles by Reichstein [R10] and Merkurjev [M13,M17] on the essential dimension.

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Univalent functions and Hardy spaces

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Abstract. This is an exposition on Hardy spaces from the viewpoint of geometric function theory. In particular, we give a brief account of both univalent analytic and univalent harmonic mappings, and discuss several results in connection with analytic and harmonic Hardy spaces.

Keywords. Integral means, Univalent functions, Harmonic functions, Growth problems, Hardy space, Baernstein's inequality, Riesz-Fejér inequality, Convex, Close-to-convex.

Subject Classification 2020: Primary: 31A05, 30C45; Secondary: 30H10.

1. Origin of Hardy Spaces

To intuitively understand the behaviour of analytic functions in the unit disk, it is often useful to look at the images of concentric circles. For example, the figure on the next page shows images of several concentric circles under the Koebe function $k(z) = z/(1-z)^2$.

One very important observation is that as r increases, the maximum value of $|k(z)|$ on the circle $|z| = r$ increases. Indeed, a stronger principle is true for not only the Koebe function, but also for any analytic function f in the unit disk $\mathbb{D} = \{z : |z| < 1\}$. Let us denote by

$$M(r) := \max_{|z|=r} |f(z)|$$

the maximum modulus of f on the circle of radius $r < 1$. Then it is known [28] that this real-valued function $M(r)$ possesses the following properties:

(i) $M(r)$ is an increasing function of r.

(ii) $\log M(r)$ is a convex function of $\log r$. This is popularly known as the *Hadamard three circle theorem*.

However, instead of the maximum modulus, a more comprehensive indication of a function's behaviour is captured by the *mean value* of $|f(z)|$ on the circle $|z| = r$, or more precisely, by the function

$$\mu(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|d\theta.$$

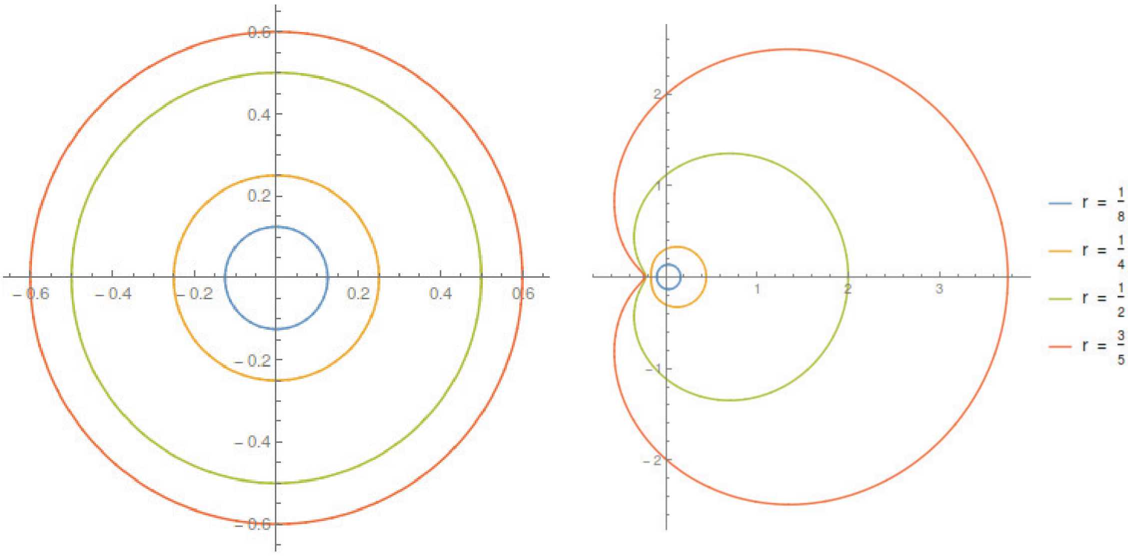

Figure 2. Circles of radius r for certain values of r and their images under the Koebe function k.

It was initially suggested to Hardy by Bohr and Landau (see [28]) that the function $\mu(r)$ exhibits property (i). However, in pursuit of the proof, Hardy surprisingly found that the function $\mu(r)$, or more generally the function

$$\mu_p(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta$$

$$\text{if } 0 < p < \infty \quad \text{and} \quad \mu_\infty(r) = M(r),$$

indeed possesses both the properties of $M(r)$. The following is widely considered as the historical starting point of the theory of Hardy spaces.

Theorem A (Hardy's Convexity Theorem). *Let f be analytic in $\mathbb{D}$ and $0 < p \le \infty$. Then*

(i) $\mu_p(r)$ is an increasing function of r.
(ii) $\log \mu_p(r)$ is a convex function of $\log r$.

Later, Hardy and Littlewood [29] discovered that the *mean values* $\mu_p(r)$ are extremely useful in understanding the boundary behaviour of an analytic function. Namely, they proved the following.

Theorem B (Hardy-Littlewood Maximal Theorem). *Let f be analytic in $\mathbb{D}$ such that $\lim_{r\to 1^-} \mu_p(r)$ exists as a finite value for some $0 < p \le \infty$, and let*

$$F(\theta) = \sup_{r<1} |f(re^{i\theta})|.$$

Then $F \in L^p[0, 2\pi]$. That is to say that F is a Lebesgue measurable function in the interval $[0, 2\pi]$, with $\|F\|_p < \infty$, where

$$\|F\|_p = \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} F^p(\theta)d\theta\right)^{\frac{1}{p}}, & 0 < p < \infty, \\ \displaystyle\sup_{\theta\in[0,2\pi]} F(\theta), & p = \infty. \end{cases}$$

The term "Hardy space" was originally used in [47] by F. Riesz, who defined these spaces in the following manner. For the forthcoming discussions, it is suitable to adopt more familiar notations.

Definition 1 (Hardy Space). *For a function f analytic in $\mathbb{D}$, the integral means $M_p(r, f)$ are defined as*

$$M_p(r, f) = \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}, & 0 < p < \infty, \\ \displaystyle\sup_{|z|=r} |f(z)|, & p = \infty. \end{cases}$$

The classical Hardy space H^p, $0 < p \le \infty$, consists of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that $M_p(r, f)$ remains bounded as $r \to 1^-$.

For example, H^∞ consists of functions that are analytic and bounded in the unit disk, and H^2 is the space of functions having the power series $\sum a_n z^n$ with $\sum |a_n|^2 < \infty$. The norm of a function $f \in H^p$ is defined as

$$\|f\|_p = \lim_{r\to 1^-} M_p(r, f).$$

For functions in the Hardy space, the boundary behaviour is of particular interest, as one can understand from the Hardy-Littlewood maximal theorem. Remarkably, Riesz established the mean convergence of an H^p-function to its boundary function.

Theorem C ([47]). *If $f \in H^p$ for some $p > 0$, then*

$$\lim_{r\to 1^-} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta = \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta$$

and

$$\lim_{r\to 1^-} \int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})|^p \, d\theta = 0,$$

where the radial limit of f exist almost everywhere on the unit circle $\mathbb{T} = \{z : |z| = 1\}$ denoted by $f(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$.

In the same paper, he also gave the following factorization formula for a function $f \in H^p$. This principle has been immensely useful in the development of the Hardy space theory.

Theorem D ([47]). *Every function $f \not\equiv 0$ of class H^p $(p > 0)$ can be factored in the form*

$$f(z) = B(z)g(z), \quad z \in \mathbb{D}, \tag{1}$$

where B is a Blaschke product consisting of the zeros of f, and g is a non-vanishing H^p-function in $\mathbb{D}$.

The study on the mean growth of analytic functions was notably continued in the paper [30] of Hardy and Littlewood, through several intricate results. For example, the following result explores the relation between the integral means of an analytic function and those of its derivative.

Theorem E ([30]). *Suppose $0 < p \le \infty$ and $\alpha > 1$, and let f be an analytic function in $\mathbb{D}$. Then*

$$M_p(r, f') = O\left(\frac{1}{(1-r)^{\alpha}}\right) \quad \text{as } r \to 1^-$$

if and only if

$$M_p(r, f) = O\left(\frac{1}{(1-r)^{\alpha-1}}\right) \quad \text{as } r \to 1^-.$$

In general terms, f' has a faster rate of growth, by a factor of $(1-r)^{-1}$, compared to the growth of f. In this context, it is worthwhile to discuss the smoothness of the boundary function. One can reasonably expect an analytic function to have a smooth extension to the boundary if the derivative grows "slowly", and vice versa. Let Λ_α $(\alpha > 0)$ be the class of functions $\varphi : \mathbb{R} \to \mathbb{C}$ satisfying a Lipschitz condition of order α: $|\varphi(x) - \varphi(y)| \le A|x - y|^\alpha$. For $\alpha > 1$, the class Λ_α consists only of constant functions. Hence one should confine interest to the case $0 < \alpha \le 1$. The next result connects the growth of the derivative of an analytic function to the smoothness of the boundary function.

Theorem F ([30]). *Let $0 < \alpha \le 1$ and f be an analytic function in $\mathbb{D}$. Then f is continuous in $\overline{\mathbb{D}}$ and $f(e^{i\theta}) \in \Lambda_\alpha$ if and only if*

$$|f'(z)| = O\left(\frac{1}{(1-r)^{1-\alpha}}\right) \quad \text{as } r = |z| \to 1.$$

On the other hand, if $f \in H^p$ $(0 < p < \infty)$, one can give a sharp estimate on the growth of $M_q(r, f)$ for any $q > p$. In fact, this statement can be expressed in a stronger form, as follows.

Theorem G ([30]). *Let f be analytic in $\mathbb{D}$ and suppose for some positive constant C,*

$$M_p(r, f) \le \frac{C}{(1-r)^{\beta}}, \quad 0 < p < \infty, \ \beta \ge 0.$$

Then there is a positive constant K, independent of f, such that

$$M_q(r, f) \le \frac{KC}{(1-r)^{\beta + \frac{1}{p} - \frac{1}{q}}}, \quad p < q \le \infty.$$

The exponent $(\beta + 1/p - 1/q)$ cannot be improved. Furthermore, if $\beta = 0$ (i.e., $f \in H^p$), then $M_q(r, f) = o\left((1-r)^{\frac{1}{q} - \frac{1}{p}}\right)$.

We refer to the books of Duren [19], Koosis [38] and Pavlović [44] for a detailed survey on integral means and Hardy spaces. Girela, Pavlović and Peláez extended Theorem E to the case $\alpha = 1$. This result is particularly useful for functions of the form $f(z) = \log(1/(1-z)^k)$, which were earlier not covered by Theorem E.

Theorem H ([25]). *If $2 < p < \infty$ and f is an analytic function in $\mathbb{D}$ such that*

$$M_p(r, f') = O\left(\frac{1}{1-r}\right) \quad \text{as } r \to 1^-,$$

then

$$M_p(r, f) = O\left(\left(\log \frac{1}{1-r}\right)^{\frac{1}{2}}\right) \quad \text{as } r \to 1^-.$$

One may be tempted to think that every analytic function has nice boundary behaviour, in the sense that the radial limit exists "almost everywhere". However, this is false, as the following example shows.

Example 1 (See [19]). *Let us consider an analytic function f of the form*

$$f(z) = \sum_{n=1}^{\infty} \epsilon_n \frac{z^{2^n}}{1 - z^{2^{n+1}}} \quad \text{where} \quad \epsilon_n = \pm 1.$$

Then for almost every sequence of signs ϵ_n, the function f has no radial limit on any set of positive measure. Consequently, $f \notin H^p$ for any $p > 0$.

However, nice boundary behaviour is observed if the considered function is *univalent*. In order to continue the discussion, let us first go through a brief overview of univalent functions.

2. An Overview of Univalent Functions

Let $\mathbb{C}$ be the complex plane. An analytic function f in a domain $D \subset \mathbb{C}$ is said to be *univalent* if it is one-to-one, i.e., $f(z_1) \neq f(z_2)$ unless $z_1 = z_2$. The function f is said to be *locally univalent* at a point $z_0 \in D$ if it is univalent in some neighbourhood of z_0. For an analytic function f, the condition $f'(z_0) \neq 0$ is necessary and sufficient for local univalence at z_0. A univalent analytic function is called a *conformal mapping* as it preserves angles and orientation.

Conformal mappings originated as means of solving problems in engineering and physics. In general, problems that can be expressed in terms of functions in $\mathbb{C}$, but exhibit complicated geometries, can be transformed into a nicer setting by an appropriate choice of conformal mapping. Given two simply connected domains $D_1, D_2 \subsetneq \mathbb{C}$, in 1851, Riemann proved that it is always possible to find a conformal mapping, which maps D_1 onto D_2. Initially Riemann's theorem defied understanding and could not find many applications, until Koebe, in 1907, gave a more complete description of these functions.

Theorem I ([37]). *Let $D \neq \mathbb{C}$ be a simply connected domain and let $z_0 \in D$. Then there exists a unique function f, analytic and univalent in D, which maps D onto the open unit disk $\mathbb{D}$ in such a way that $f(z_0) = 0$ and $f'(z_0) > 0$.*

By virtue of this strong version of the Riemann mapping theorem, numerous problems about simply connected domains can be reduced to the special case of the unit disk. In particular, the study of univalent functions between two arbitrary simply connected domains is equivalent to the study of univalent functions from $\mathbb{D}$ onto any simply connected domain $D \neq \mathbb{C}$. Also, the normalization conditions $f(0) = 0 = f'(0) - 1$ prove to be helpful, and do not affect any result pertaining to univalent functions. We let $\mathcal{S}$ denote the family of analytic, univalent and normalized functions defined in $\mathbb{D}$. Thus, a function f in $\mathcal{S}$ has the power series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{2}$$

It is well-known that $\mathcal{S}$ is compact with respect to the topology of uniform convergence on compact subsets of $\mathbb{D}$. The Koebe function

$$k(z) = z/(1-z)^2 = z + \sum_{n=2}^{\infty} n z^n,$$

which maps $\mathbb{D}$ onto the whole complex plane minus the slit $(-\infty, -1/4]$, is extremal for many problems in the class $\mathcal{S}$. In 1916, Bieberbach [8] started the problem on coefficient bounds for functions $f \in \mathcal{S}$ and observed that $|a_2| \leq 2$, while the equality occurs only for the Koebe function and its rotations. This led him to make the following conjecture.

Conjecture A ([8]). *If $f \in \mathcal{S}$ is any function of the form (2), then $|a_n| \leq n$ for all $n \geq 2$. Furthermore, $|a_n| = n$ for all n if and only if f is the Koebe function k, or its rotations.*

In 1925, the first significant progress on the conjecture was made by Littlewood [41], who showed that $|a_n| < en$, ensuring that the Bieberbach conjecture has the correct order of magnitude. Over the years, the constant e was successively replaced by a string of smaller constants, although a complete proof remained elusive. Finally, it was de Branges [17] who settled the conjecture affirmatively in 1985, i.e., almost 70 years after its origin.

Failure to settle the Bieberbach conjecture for a long time led to the origin and development of several subclasses of $\mathcal{S}$. A nice subclass, denoted by $\mathcal{K}$, consists of the functions that map $\mathbb{D}$ onto a convex domain. This geometric subclass can be neatly described through the following analytic characterization.

Theorem J ([20, Theorem 2.11]). *Let $f \in \mathcal{S}$. Then $f \in \mathcal{K}$ if and only if*

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D}.$$

A function $f \in \mathcal{K}$ is called a *convex function*. Curiously, for these functions the coefficient bound $|a_n| \le 1$ holds, with equality occurring for the function

$$l(z) = z/(1-z) = \sum_{n=1}^{\infty} z^n,$$

which maps $\mathbb{D}$ onto the half-plane $\mathrm{Re}\{w\} > -1/2$. Perhaps the most interesting geometric subclass of $\mathcal{S}$ is the family C of functions which map $\mathbb{D}$ onto a close-to-convex domain, i.e., a domain whose complement can be expressed as a union of non-intersecting half-lines. Functions in C are called *close-to-convex*. The following analytic characterization of close-to-convex functions, due to Kaplan, is very useful.

Theorem K ([35]). *Let f be analytic and locally univalent in $\mathbb{D}$. Then f is close-to-convex if and only if*

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) d\theta > -\pi, \quad z = re^{i\theta},$$

for every r $(0 < r < 1)$ and for every pair of real numbers θ_1 and θ_2 with $\theta_1 < \theta_2$.

It is easy to see that $\mathcal{K} \subsetneq C$, as the Koebe function and its rotations are in C, but not in $\mathcal{K}$. A detailed study of the class $\mathcal{S}$ and its major subclasses can be found in the monographs of Duren [20], Goodman [26,27] and Pommerenke [45].

3. Hardy Spaces in the Study of Univalent Functions

Let us first try to intuitively understand the Bieberbach conjecture from the perspective of area of the range $f(\mathbb{D}_r)$, for $f \in \mathcal{S}$ where $\mathbb{D}_r = \{z : |z| < r\}$. If D is a measurable set in the complex plane, then the area of D is given by

$$\mathrm{Area}(D) = \iint_D dx\, dy.$$

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a conformal mapping of the unit disk $\mathbb{D}$ onto a domain Ω, then by change of variables formula, the area of the image $\Omega_r = f(\mathbb{D}_r)$ is given by

$$\begin{aligned}
\mathrm{Area}(\Omega_r) &= \iint_{\mathbb{D}_r} |f'(z)|^2\, dx\, dy \\
&= \int_0^r \int_0^{2\pi} \rho |f'(\rho e^{i\theta})|^2\, d\theta\, d\rho \\
&= \int_0^r \int_0^{2\pi} \rho \left(\sum_{n=1}^{\infty} n a_n \rho^{n-1} e^{i(n-1)\theta}\right) \\
&\qquad \times \left(\sum_{n=1}^{\infty} n \overline{a_n} \rho^{n-1} e^{-i(n-1)\theta}\right) d\theta\, d\rho \\
&= \int_0^r \left(\sum_{n=1}^{\infty} 2\pi n^2 |a_n|^2 \rho^{2n-1}\right) d\rho \\
&= \pi \sum_{n=1}^{\infty} n |a_n|^2 r^{2n}
\end{aligned}$$

Thus, it is clear that the area covered by the image of a disk $\mathbb{D}_r$ under conformal mappings increases as the absolute values of the coefficients a_n increase. Therefore, $|a_n|$ should be the highest for a function that covers the maximum possible area in the complex plane $\mathbb{C}$. Since the image of $\mathbb{D}_r$ under the Koebe function k expand and cover $\mathbb{C} \backslash (-\infty, -1/4]$ as $r \to 1^-$, it is reasonable to expect k to possess the maximum modulus value of coefficients, which is the Bieberbach conjecture. The estimate for the upper bound of the following integrals

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p\, \mathrm{d}\theta \ \text{ and } \ \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^p\, \mathrm{d}\theta$$

got special attention for $p = -2, 1, 2$, as these have applications in the context of free boundary problems in fluid dynamics. Please note that in the above case, when $p = -2$, the integrand $|f'(re^{i\theta})|^{-2}$ should be interpreted as $(1/|f'(re^{i\theta})|)^2$. This allows us to apply results from Hardy Spaces.

Historically, the integral mean $M_1(r, f)$ is closely related to the Bieberbach conjecture. The primary tool in Littlewood's proof of $|a_n| < en$ is the inequality $M_1(r, f) \le r/(1-r)$ for any $f \in \mathcal{S}$. If this estimate can be improved to

$$M_1(r, f) \le r/(1-r^2) = M_1(r, k),$$

the same argument leads to the much better bound $|a_n| < (e/2)n$. This gave rise to the natural interest to find the sharp upper bound for $M_1(r, f)$, or more generally, for $M_p(r, f)$, $0 < p < \infty$.

In 1951, Bazilevich [6] produced a partial approach to this problem for the cases $p = 1, 2$. He showed that

$$M_p(r, f) < M_p(r, k) + C_p, \quad p = 1, 2,$$

where C_p is a constant, given explicitly, that does not depend on f. Years later, in 1974, Baernstein [4] introduced radically new methods to prove that

$$M_p(r, f) \le M_p(r, k), \quad 0 < p < \infty,$$

i.e., among the functions of class $\mathcal{S}$, the Koebe function has the largest integral mean. Indeed, Baernstein obtained a much more general inequality for convex functions, the proof of which involves a curious maximal function, namely the star-function.

Theorem L ([4]). *If $f \in \mathcal{S}$ and $\Phi(x)$ is a convex nondecreasing function on $(-\infty, \infty)$, then*

$$\int_{-\pi}^{\pi} \Phi\left(\log|f(re^{i\theta})|\right) d\theta \le \int_{-\pi}^{\pi} \Phi\left(\log|k(re^{i\theta})|\right) d\theta,$$

where k is the Koebe function. For the choice $\Phi(x) = e^{px}$, one gets

$$M_p(r, f) \le M_p(r, k), \quad 0 < p < \infty.$$

Leung [39] and Brown [9] extended Baernstein's theorem to derivatives of certain subclasses of univalent functions. Extremal problems of this type are widely studied in the literature (see, for example, [5,20,23,24]) and play a central role in the growth of univalent functions. In [23], Girela obtained Baernstein type results for the functions $\log(f(z)/z)$. These functions appear in the definition of *logarithmic coefficients* γ_n of a function $f \in \mathcal{S}$:

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n,$$

which were instrumental in de Branges' proof of the Bieberbach conjecture (see [17]). Girela's work readily led to the sharp inequality

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \le \frac{\pi^2}{6},$$

an important estimate obtained earlier by Duren and Leung [22]. Interestingly, Girela proved the following extremal result for close-to-convex functions.

Theorem M ([23]). *Let $f \in \mathcal{S}$ be close-to-convex and $0 < p \le 2$. Then*

$$M_p(r, \log f') \le M_p(r, \log k'),$$

where k is the Koebe function.

Another important result on the growth of univalent functions is their membership in the Hardy space, as follows.

Theorem N ([19, Theorem 3.16]). *If f is analytic and univalent in $\mathbb{D}$, then $f \in H^p$ for all $p < 1/2$.*

This result has numerous implications. For example, it ensures that a conformal mapping of $\mathbb{D}$ onto any arbitrary simply connected domain, regardless of how complicated the boundary is, automatically has a radial limit in almost every direction. It also asserts that as a member of H^p, every univalent function f has the factorization (1), where it is obvious that the Blaschke product B has at most one factor. The Koebe function k, which does not belong to $H^{1/2}$, shows that the range $p < 1/2$ is best possible. However, it is known that every convex function $f \in \mathcal{K}$ is of class H^p for all $p < 1$. Again, the range is sharp as the function $l(z) = z/(1-z) \notin H^1$.

Finally, we recall a result due to Riesz and Fejér concerning Hardy Spaces, which also has intriguing applications to univalent functions.

Theorem O ([19, Theorem 3.13]). *If $f \in H^p$ $(0 < p < \infty)$, then the integral of $|f(x)|^p$ along the segment $-1 \le x \le 1$ converges, and*

$$\int_{-1}^{1} |f(x)|^p dx \le \frac{1}{2}\int_0^{2\pi} |f(e^{i\theta})|^p d\theta. \tag{3}$$

The constant $1/2$ is best possible.

This theorem has a nice geometric description for univalent functions: *if the unit disk is mapped conformally onto the interior of a rectifiable Jordan curve C, the length of the image of any diameter cannot exceed half the length of C.* Over the years there have been several generalizations of this result. Beckenbach [7] notably proved that the same inequality remains true if in place of $|f|^p$, we consider a positive function whose logarithm is subharmonic. Some generalizations of Theorem O under weaker regularity assumptions, or in different spaces, may be found in [11,33,3,18] and the references therein.

4. Univalent harmonic functions

In what follows, we reserve the term "harmonic function" to mean complex-valued harmonic function, unless otherwise specified. Also, we use the terms "harmonic mapping" and "harmonic function" interchangeably, as this is customary in recent literature.

Univalent harmonic functions can be thought of as a natural generalization of conformal mappings. However, unlike conformal mappings, these functions are not at all determined (up to normalization) by their image domains. Univalent harmonic functions in $\mathbb{C}$ have traditionally appeared in the description of minimal surfaces. For instance, in 1952, Heinz [31] studied the Gaussian curvature of non-parametric minimal surfaces over $\mathbb{D}$ by making use of such functions. After the emergence of the seminal paper [13] of Clunie and Sheil-Small, univalent harmonic functions generated interest more from a function theoretic point of view. This approach had a clear advantage: the functions could now be treated with elegant function theoretic methods that were earlier not in use for similar problems, while the results could still be connected to the theory of minimal surfaces. Also, it was observed that univalent harmonic functions with nice geometric properties has particular importance in the study of minimal surfaces, thereby making the geometric subclasses of such functions quite interesting.

A complex-valued function $f = u + iv$ is harmonic in the unit disk if u and v are real-valued harmonic functions in $\mathbb{D}$. Every such function has a unique representation $f = h + \bar{g}$, where h, g are analytic functions in $\mathbb{D}$ with $g(0) = 0$. The function h is said to be the *analytic part*, and g the *co-analytic part*, of f. Thus, harmonic functions exhibit a two-folded series structure: one is a power series in z, and the other being a power series in $\bar{z}$.

It is clear that every analytic function is indeed harmonic, but the converse is not true. In particular, the functions u and v need not satisfy the Cauchy-Riemann equations. This relaxation significantly affects the behaviour of harmonic functions. In contrast to analytic functions, the composition, product, reciprocal and inverse of harmonic functions need not be harmonic. Surprisingly though, it is true that if f is harmonic and g is analytic, then the composition $f \circ g$, suitably defined, is harmonic. This, together with the Riemann mapping theorem and the following well-known result of Radó, reduce the study of univalent harmonic mappings in any arbitrary simply connected domain $D \neq \mathbb{C}$ to the study of such mappings in $\mathbb{D}$.

Theorem P ([21, p. 24]). *There is no univalent harmonic function which maps $\mathbb{D}$ onto $\mathbb{C}$.*

As mentioned earlier, an analytic function is locally univalent at a point z_0 if and only if $f'(z_0) \neq 0$. Since the Jacobian $J_f(z)$ equals $|f'(z)|^2$ for an analytic function f, it means that every locally univalent analytic function has a non-vanishing Jacobian. Lewy [40] showed that the same principle remains true for planar harmonic mappings.

Theorem Q. *Let $f = h + \bar{g}$ be a harmonic function defined in a domain $D \subset \mathbb{C}$. If f is locally univalent at $z_0 \in D$, then*

$$J_f(z_0) = |h'(z_0)|^2 - |g'(z_0)|^2 \neq 0.$$

The Jacobian of a locally univalent harmonic function, since continuous, has the same sign throughout a domain. The function f is said to be *sense-preserving* (or, *orientation-preserving*) in D if $J_f(z) > 0$ for all $z \in D$, and to be *sense-reversing* if $J_f(z) < 0$ for every $z \in D$. If f is sense-reversing, then $\bar{f}$ is sense-preserving, so one may confine interest to sense-preserving harmonic functions, without any loss of generality. In the context of the unit disk, a harmonic function $f = h + \bar{g}$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if the inequality $|h'(z)| > |g'(z)|$ holds for every $z \in \mathbb{D}$. Associated with every such function is the *second complex dilatation* $w(z) = g'(z)/h'(z)$, which satisfies $|w(z)| < 1$ on $\mathbb{D}$.

Let S_H be the class of all sense-preserving univalent harmonic functions $f = h + \bar{g}$ in $\mathbb{D}$ normalized by $h(0) = g(0) = h'(0) - 1 = 0$. Thus, each function $f = h + \bar{g} \in S_H$ admits the representation

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n.$$

It is known that S_H is a normal family, but not compact (See [13]). For instance, the functions $f_n(z) = z + (n/(n+1))\bar{z}$ are in S_H, but as $n \to \infty$, $f_n(z) \to 2\mathrm{Re}\,(z)$, which is not univalent. Therefore, to study extremal problems, e.g., the upper bounds of coefficients, it is often more convenient to work with the compact normal family $S_H^0 = \{f = h + \bar{g} \in S_H : g'(0) = 0\}$, which is in correspondence with the class S_H. If $f \in S_H$, then

$|b_1| < 1$ and the function

$$f_0 = \frac{f - \overline{b_1 f}}{1 - |b_1|^2}$$

is in S_H^0. Similarly, for $f_0 \in S_H^0$ and for each fixed complex number b_1 with $|b_1| < 1$, the function $f = f_0 + \overline{b_1 f_0}$ belongs to S_H. Analogous to the geometric subclasses of $\mathcal{S}$, one can define various subclasses of S_H. Let K_H and C_H be the subclasses of S_H consisting of harmonic mappings onto convex and close-to-convex domains, respectively, and let $K_H^0 = K_H \cap S_H^0$ and $C_H^0 = C_H \cap S_H^0$ be the corresponding compact classes. Two leading examples of univalent harmonic functions are

$$L(z) = H_1(z) + \overline{G_1(z)} = \left(\frac{z - \frac{1}{2}z^2}{(1-z)^2}\right) + \overline{\left(\frac{-\frac{1}{2}z^2}{(1-z)^2}\right)}$$

which maps the unit disk onto the half-plane $\operatorname{Re}\{w\} > -1/2$, and the harmonic Koebe function

$$K(z) = H_2(z) + \overline{G_2(z)} = \left(\frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}\right) + \overline{\left(\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}\right)}$$

which maps $\mathbb{D}$ onto the entire plane minus the real interval $(-\infty, -1/6]$. It is easy to see that $L \in K_H^0$ and $K \in C_H^0$. Interestingly, these functions originated through a method of *shear construction* due to Clunie and Sheil-Small, which is the most well-known tool for constructing univalent harmonic mappings in $\mathbb{D}$ (with prescribed dilatation).

Theorem R ([13, Theorem 5.3]). *Let $f = h + \bar{g}$ be a locally univalent harmonic function in $\mathbb{D}$. Then f is univalent and its range is convex in the horizontal direction (resp. vertical direction) if and only if $h - g$ (resp. $h + g$) has the same properties.*

Theorem R makes it possible to construct univalent harmonic functions convex in the horizontal direction, by “shearing" (i.e., stretching and translating) the range of a given univalent analytic function in the horizontal direction. The necessary steps are as follows.

(i) Choose $h - g = \phi$, where $\phi \in \mathcal{S}$ maps $\mathbb{D}$ onto a domain convex in the horizontal direction.

(ii) Choose an analytic function w in $\mathbb{D}$ with $|w(z)| < 1$.

(iii) Solve the relations

$$h' - g' = \phi' \quad \text{and} \quad wh' = g'$$

to find h and g.

(iv) The solutions are

$$h(z) = \int_0^z \frac{\varphi'(\zeta)}{1 - \omega(\zeta)} d\zeta \quad \text{and} \quad g(z) = h(z) - \phi(z).$$

(v) Then the desired harmonic function is

$$f(z) = h(z) + \overline{g(z)} = 2\operatorname{Re}(h(z)) - \overline{\phi(z)}.$$

Similarly, one can choose $h + g = \varphi$, where $\varphi \in \mathcal{S}$ maps $\mathbb{D}$ onto a domain convex in the vertical direction, and follow the above steps to construct univalent harmonic functions convex in the vertical direction. Using this method, the harmonic half-plane mapping L arises through the choices

$$h(z) + g(z) = l(z) = z/(1-z) \quad \text{and} \quad w(z) = -z,$$

while the harmonic Koebe function is obtained by choosing

$$h(z) - g(z) = k(z) = z/(1-z)^2 \quad \text{and} \quad w(z) = z.$$

More details on univalent harmonic functions can be found in the paper of Clunie and Sheil-Small [13], as well as in the monograph of Duren [21] and the expository article of Bshouty and Hengartner [10].

The following harmonic analogue of the Bieberbach conjecture due to Clunie and Sheil-Small has been the primary motivation behind the theory of univalent harmonic functions.

Conjecture B ([13]). *Suppose $f = h + \bar{g} \in S_H^0$, with $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=2}^{\infty} b_n z^n$. Then for all $n \geq 2$,*

$$|a_n| \leq \frac{(n+1)(2n+1)}{6},$$

$$|b_n| \leq \frac{(n-1)(2n-1)}{6}, \quad \text{and} \quad ||a_n| - |b_n|| \leq n.$$

The bounds are attained for the harmonic Koebe function K.

The conjecture has been verified for a number of subclasses of S_H^0, see [13,48]. Most notably, Wang, Liang and Zhang [50] verified the conjecture for the class C_H^0. For the whole class S_H^0, the inequality $|b_2| \leq 1/2$ has been established, but the problem remains vastly open, even for $|a_2|$. To this end, the latest known bound is $|a_2| < 21$, due to Abu Muhanna, Ali and Ponnusamy [1]. It is pertinent to mention that for functions $f \in K_H^0$, the improved bounds

$$|a_n| \leq \frac{n+1}{2}, \quad |b_n| \leq \frac{n-1}{2}, \quad \text{and} \quad ||a_n| - |b_n|| \leq 1$$

are known. Equality occurs for the half-plane mapping L.

The representation $f = h + \bar{g}$, in view of the rich theory of Hardy spaces of analytic functions, led to considerable interest in the boundary behaviour of planar harmonic mappings. Analogous to the H^p spaces, the harmonic Hardy spaces h^p are defined as the class of harmonic functions f in $\mathbb{D}$ which satisfy

$$\|f\|_p = \sup_{0 \le r < 1} M_p(r, f) < \infty.$$

When a harmonic function $f \in h^p$ for some $p > 1$, then $\sup_{0 \le r < 1} M_p(r, f) = \lim_{r \to 1^-} M_p(r, f)$. In [2], Abu-Muhanna and Lyzzaik showed that there exists a universal $p > 0$ such that every $f \in S_H$ belongs to the class h^p. This implies that every univalent harmonic function in $\mathbb{D}$ has a finite radial limit in almost every direction. Later, Nowak [43] improved the results of Abu-Muhanna and Lyzziak, and obtained sharp estimates for $p > 0$ such that the classes K_H and C_H are contained in h^p. However, the exact range of $p > 0$ for the whole class S_H remains unknown.

5. Recent Results for Harmonic Functions

5.1 Growth of harmonic functions in the unit disk

In a relatively recent development, Chen, Ponnusamy and Wang [12] observed that Theorem H remains valid for harmonic functions as well. Let us state the result.

Theorem S. *Suppose $p > 2$ and f is a harmonic function in $\mathbb{D}$. Let $\nabla f = (f_z, f_{\bar{z}})$ and $|\nabla f| = \left(|f_z|^2 + |f_{\bar{z}}|^2\right)^{\frac{1}{2}}$. If*

$$M_p(r, \nabla f) = O\left(\frac{1}{1-r}\right) \quad as\ r \to 1^-,$$

then

$$M_p(r, f) = O\left(\left(\log \frac{1}{1-r}\right)^{\frac{1}{2}}\right) \quad as\ r \to 1^-.$$

For an analytic function f, it is obvious that $|\nabla f(z)| = |f'(z)|$. Therefore, this result particularly contains the result of Girela, Pavlović and Peláez. While Theorem S extends Theorem H to harmonic functions, the harmonic analogues of the more fundamental Theorems E–G were not known. Therefore, one is naturally intrigued by the question: *to what extent are the growth results for analytic functions valid for harmonic functions?* Very recently, the present authors [15] have showed that these results indeed hold in the setting of harmonic functions in $\mathbb{D}$, leading to the understanding that analytic and harmonic functions behave alike in regards to growth.

Theorem 1 ([15]). *Let $1 \le p < \infty$ and $\alpha > 1$. If f is a harmonic function in $\mathbb{D}$, then*

$$M_p(r, \nabla f) = O\left(\frac{1}{(1-r)^{\alpha}}\right) \quad as\ r \to 1^-$$

if and only if

$$M_p(r, f) = O\left(\frac{1}{(1-r)^{\alpha-1}}\right) \quad as\ r \to 1^-.$$

Theorem 2 ([15]). *Let $0 < \alpha < 1$ and f be a harmonic function in $\mathbb{D}$. Then f is continuous in $\overline{\mathbb{D}}$ and $f(e^{i\theta}) \in \Lambda_\alpha$ if and only if*

$$|\nabla f(z)| = O\left(\frac{1}{(1-r)^{1-\alpha}}\right) \quad as\ r = |z| \to 1.$$

Theorem 3 ([15]). *Let f be harmonic in $\mathbb{D}$ and suppose for some positive constant C,*

$$M_p(r, f) \le \frac{C}{(1-r)^{\beta}}, \quad 1 \le p < \infty,\ \beta \ge 0.$$

Then there is a positive constant K independent of f such that

$$M_q(r, f) \le \frac{KC}{(1-r)^{\beta + \frac{1}{p} - \frac{1}{q}}}, \quad p < q \le \infty. \tag{4}$$

The exponent $(\beta + 1/p - 1/q)$ is best possible. Furthermore, if $\beta = 0$ (i.e., $f \in h^p$), then $M_q(r, f) = o\left((1-r)^{\frac{1}{q} - \frac{1}{p}}\right)$.

Closely related to the coefficient problem for functions in S_H is the mean growth of these functions, in the sense that the study of integral means enables one to estimate the Taylor series coefficients of the corresponding analytic and co-analytic parts. The following result gives an order of growth for the integral mean of a function $f \in S_H$, and as a consequence, produces a coefficient estimate.

Theorem 4 ([15]). *Suppose $f = h + \bar{g} \in S_H$ with $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$. Let $\alpha = \sup_{f \in S_H} |a_2|$. Then for every $\epsilon > 0$,*

$$M_p(r, f) = O\left(\frac{1}{(1-r)^{k(p)+\epsilon}}\right) \quad (1 \le p < \infty),$$

where $k(p) = \sqrt{\alpha^2 - \frac{1}{p} + \frac{1}{4p^2}} - \frac{1}{2p}$. Consequently,

$$|a_n| = O(n^{\alpha - \frac{1}{2}}), \quad |b_n| = O(n^{\alpha - \frac{1}{2}}).$$

This coefficient bound is an improvement on an earlier estimate by Starkov [49, Lemma 2] which involves n^{α}.

5.2 Univalent harmonic functions in the Hardy space

A central problem pertaining to the growth of univalent harmonic mappings is to determine the range of $p > 0$ so that a function f belongs to the harmonic Hardy space h^p. The initial developments in this direction were due to Abu-Muhanna and Lyzzaik [2], who proved that $f \in h^p$ for $p < 1/(2\alpha + 2)^2$. Bshouty and Hengartner [10] proposed to find the exact range of $p > 0$ for which $f \in h^p$. In [43], Nowak improved the range to $p < 1/\alpha^2$, and obtained the sharp results that $f \in h^p$ for $p < 1/2$ (resp. $p < 1/3$) whenever f is a convex (resp. close-to-convex) harmonic function. These observations led her to conjecture that if $f \in S_H$, then $f \in h^p$ for $p < 1/\alpha$. The conjecture seems challenging, and in [46] Ponnusamy, Qiao and Wang verified it for harmonic quasiconformal mappings.

In [16], the present authors give a relation between $M_p(r, f)$ and $M_p(r, h')$, which naturally allows one to check the boundedness of $\|f\|_p$ whenever h' behaves "nicely". As it turns out, this can be achieved by placing the simple restriction that h' takes no value infinitely often (in other words, h' has *finite valency*). Thus, Nowak's conjecture is verified for functions of this type in S_H. Indeed, the result is true for a more general class of functions. Let us recall that a family $\mathcal{L}$ of harmonic functions in $\mathbb{D}$ is said to be *linear invariant* (see [48]) if for every $f = h + \bar{g} \in \mathcal{L}$, the functions

$$T_\varphi(f(z)) = \frac{f(\varphi(z)) - f(\varphi(0))}{\varphi'(0)h'(\varphi(0))}, \quad \varphi \in \mathrm{Aut}(\mathbb{D}),$$

belong to $\mathcal{L}$, where $\mathrm{Aut}(\mathbb{D})$ denotes the set of analytic automorphisms of $\mathbb{D}$. The result does not require univalence, and holds for any linear invariant class $\mathcal{H}$ of locally univalent and sense-preserving harmonic functions (with usual normalizations), for which $\alpha(\mathcal{H}) = \sup_{f\in\mathcal{H}} |a_2|$ is finite. Let us preserve the notation $\mathcal{H}$ to mean any such class of locally univalent harmonic functions.

Theorem 5 ([16]). *Let $f = h+\bar{g} \in S_H$ be such that h' takes no value infinitely often. Then $f \in h^p$ for $p < 1/\alpha$. If $f \in \mathcal{H}$ and h' has finite valency, then $f \in h^p$ for $p < 1/\alpha(\mathcal{H})$.*

Also, the following coefficient bound is obtained for these functions.

Theorem 6 ([16]). *Suppose $f = h + \bar{g} \in S_H$ has series representation as in* Theorem 4 *and h' takes no value infinitely often. Then $|a_n|$ and $|b_n|$ are $O(n^{\alpha-1})$, $n = 2, 3, 4, \ldots$. For $f \in \mathcal{H}$ with h' having finite valency, $|a_n|$ and $|b_n|$ are $O(n^{\alpha(\mathcal{H})-1})$.*

This coefficient estimate for $f \in S_H$ is in a sense best possible. The conjectured value of α is 3. Given this, Theorem 6 asserts that $|a_n|$ and $|b_n|$ are $O(n^2)$, which is the same order as in the harmonic analogue of the Bieberbach conjecture (Conjecture B).

The problem, even without the assumption of finite valency, is explored in another direction to produce a very interesting result. For $f = h + \bar{g} \in S_H$, one can show that

$$\left|\frac{h''(z)}{h'(z)}\right| \le \frac{C}{1 - |z|} \quad (z \in \mathbb{D}),$$

for some positive constant C. However, this is the extreme bound on h''/h' that a function $f = h + \bar{g} \in S_H$ can possess. In general, it is reasonable to expect a large subclass of S_H to have a slightly restricted growth, or more precisely, to exhibit the bound

$$\left|\frac{h''(z)}{h'(z)}\right| \le \frac{C}{(1 - |z|)^\beta} \quad (0 \le \beta < 1).$$

This growth condition on h''/h' yields the following result on the membership of univalent and locally univalent harmonic functions in the Hardy space.

Theorem 7 ([16]). *Let $f = h + \bar{g} \in S_H$ be such that*

$$\left|\frac{h''(z)}{h'(z)}\right| \le \frac{C}{(1 - |z|)^\beta}, \tag{5}$$

for some β with $0 \le \beta < 1$. Then $f \in h^p$ for $p < 2(1 - \beta)/\alpha$. Analogously, if $f = h + \bar{g} \in \mathcal{H}$ satisfies the growth estimate (5), then $f \in h^p$ for $p < 2(1 - \beta)/\alpha(\mathcal{H})$.

5.3 Baernstein type extremal results

While Baernstein type problems are widely studied for analytic functions, the same for harmonic functions remained unexplored. Recently in [15], the present authors produced Baernstein type results for the geometric subclasses of harmonic functions. Let us recall the (compact) classes K_H^0 and C_H^0 of convex and close-to-convex harmonic functions, respectively, and also the half-plane mapping $L(z) = H_1(z) + \overline{G_1(z)}$ and the harmonic Koebe function $K(z) = H_2(z) + \overline{G_2(z)}$. We know that $L \in K_H^0$ and $K \in C_H^0$. As it turns out, these functions play the extremal role in Baernstein type inequalities for the respective classes.

Theorem 8 ([15]). *Let $0 < p < \infty$. If $f = h + \bar{g} \in K_H^0$ and $\Phi(x)$ is a convex nondecreasing function on $(-\infty, \infty)$, then*

$$\int_{-\pi}^{\pi} \Phi\left(\log|h'(re^{i\theta})|\right) d\theta \le \int_{-\pi}^{\pi} \Phi\left(\log|H_1'(re^{i\theta})|\right) d\theta,$$

$$\int_{-\pi}^{\pi} \Phi\left(\log|g'(re^{i\theta})|\right) d\theta \le \int_{-\pi}^{\pi} \Phi\left(\log|G_1'(re^{i\theta})|\right) d\theta.$$

Consequently,

$$M_p(r, h') \le M_p(r, H_1') \quad \text{and} \quad M_p(r, g') \le M_p(r, G_1').$$

Since $L = H_1 + \overline{G_1} \in K_H^0$, these inequalities are sharp.

Theorem 9 ([15]). *Let $0 < p < \infty$. If $f = h + \bar{g} \in C_H^0$ and $\Phi(x)$ is a convex nondecreasing function on $(-\infty, \infty)$, then*

$$\int_{-\pi}^{\pi} \Phi\left(\log|h'(re^{i\theta})|\right) d\theta \le \int_{-\pi}^{\pi} \Phi\left(\log|H_2'(re^{i\theta})|\right) d\theta,$$

$$\int_{-\pi}^{\pi} \Phi\left(\log|g'(re^{i\theta})|\right) d\theta \le \int_{-\pi}^{\pi} \Phi\left(\log|G_2'(re^{i\theta})|\right) d\theta.$$

Consequently,

$$M_p(r, h') \le M_p(r, H_2') \quad \text{and} \quad M_p(r, g') \le M_p(r, G_2').$$

Since $K = H_2 + \overline{G_2} \in C_H^0$, these inequalities are sharp.

These results have nice geometric appeal. For $0 < r < 1$, the length of the curve $C(r) = \{f(re^{i\theta}) = h(re^{i\theta}) + \overline{g(re^{i\theta})} : \theta \in [0, 2\pi)\}$, counting multiplicity, is defined by

$$\begin{aligned}\mathcal{L}_f(r) &= \int_0^{2\pi} |df(re^{i\theta})| \\ &= r\int_0^{2\pi} \left|h'(re^{i\theta}) - e^{-2i\theta}\overline{g'(re^{i\theta})}\right| d\theta.\end{aligned}$$

In case of sense-preserving harmonic mappings, we get

$$\mathcal{L}_f(r) \le r(1+r)\int_0^{2\pi} |h'(re^{i\theta})| d\theta = 2\pi r(1+r)M_1(r, h').$$

With this observation, one easily gets the following corollaries.

Corollary 1. *If $f = h + \bar{g} \in K_H^0$, then $\mathcal{L}_f(r) \le (1+r)\mathcal{L}_{H_1}(r)$.*

Corollary 2. *If $f = h + \bar{g} \in C_H^0$, then $\mathcal{L}_f(r) \le (1+r)\mathcal{L}_{H_2}(r)$.*

Theorems 8 and 9 also lead to integral mean estimates for functions in the respective classes.

Theorem 10 ([15]). *The following bounds hold:*

(i) If $1 \le p < \infty$, then

$$M_p(r, f) \le B_p \int_0^r (1 + s^p)^{\frac{1}{p}} M_p(s, H_1')ds \quad (f \in K_H^0),$$

$$M_p(r, f) \le B_p \int_0^r (1 + s^p)^{\frac{1}{p}} M_p(s, H_2')ds \quad (f \in C_H^0),$$

where

$$B_p = \begin{cases} \sqrt{2}, & 1 \le p \le 2, \\ 2^{1-\frac{1}{p}}, & p > 2. \end{cases}$$

(ii) For $0 < p < 1$, we have

$$M_p^p(r, f) \le C_1 \int_0^r (r - s)^{p-1} M_p^p(s, H_1')ds \quad (f \in K_H^0),$$

$$M_p^p(r, f) \le C_2 \int_0^r (r - s)^{p-1} M_p^p(s, H_2')ds \quad (f \in C_H^0),$$

where C_1 and C_2 are absolute constants.

In [16], a harmonic analogue of Girela's result (Theorem M) is obtained. Namely, it is proved that Theorem M remains true for the functions $\log(h' + cg')$, whenever $f = h + \bar{g}$ is a close-to-convex harmonic function and for any constant $c \in \mathbb{D}$.

Theorem 11. *Suppose $0 < p \le 2$ and $f = h + \bar{g} \in C_H^0$. Then for any constant $c \in \mathbb{D}$, we have*

$$M_p(r, \log(h' + cg')) \le M_p(r, \log(H_2' + G_2')).$$

The bound is sharp.

To explore the logarithmic coefficients in the setting of a harmonic mapping $f = h + \bar{g}$, it is not feasible to consider $f(z)/z$ (i.e., $(h(z) + \overline{g(z)})/z$) as this function need not be harmonic, neither is the logarithm of a harmonic function defined in the literature. One can not consider the function $(h(z) + g(z))/z$ either, since $h(z) + g(z)$ may have zeros at points other than the origin. Therefore, proceeding along the line of Theorem 11, the functions $\log(h' + cg')$ ($c \in \mathbb{D}$ constant) seem to be the most natural choice. Suppose $\log(h'(z) + cg'(z)) = \sum_{n=1}^{\infty} \lambda_n z^n$. Then Theorem 11 has the following implication.

Corollary 3. *Let $f = h + \bar{g} \in C_H^0$. Then we have the sharp inequality*

$$\sum_{n=1}^{\infty} |\lambda_n|^2 \le \frac{14\pi^2}{3}.$$

5.4 Riesz-Fejér type inequalities

Long back, the study of Riesz-Fejér inequalities for complex-valued harmonic functions was initiated by Riesz and Zygmund, who were motivated to find a similar geometry for harmonic functions.

Theorem T (Riesz-Zygmund). *If f is a complex-valued harmonic functions in $\mathbb{D}$ such that $\frac{\partial}{\partial\theta} f(re^{i\theta}) \in h^1$, then*

$$\int_{-1}^{1} \left| \frac{\partial}{\partial r} f(re^{it}) \right| dr \leq \frac{1}{2} \sup_{0<r<1} \int_0^{2\pi} \left| \frac{\partial}{\partial\theta} f(re^{i\theta}) \right| d\theta.$$

The constant $1/2$ is sharp.

This inequality gives rise to a geometry analogous to the classical result of Riesz and Fejér: *If f is a harmonic diffeomorphism from $\mathbb{D}$ onto the interior of a rectifiable Jordan curve C, then the image of any diameter has length at most half the length of C.* In recent times, Kalaj [34] deduced the following result using a method of plurisubharmonic functions, originally due to Hollenbeck and Verbitsky [32].

Theorem U ([34]). *Let $1 < p < \infty$. Suppose $f = h + \overline{g} \in h^p$ with* $\mathrm{Re}\,(h(0)g(0)) = 0$. *Then*

$$\int_{\mathbb{T}} (|h(z)|^2 + |g(z)|^2)^{\frac{p}{2}} |dz| \leq \frac{1}{(1 - |\cos\frac{\pi}{p}|)^{\frac{p}{2}}} \int_{\mathbb{T}} |f(z)|^p |dz|.$$

The inequality is sharp.

In the breakthrough paper [36], Kayumov, Ponnusamy and Sairam Kaliraj managed to obtain a complete harmonic analogue of the Riesz-Fejér inequality for the harmonic Hardy space h^p, $p \in (1, 2]$.

Theorem V ([36]). *If $f \in h^p$ for $p \in (1, 2]$, then*

$$\int_{-1}^{1} |f(xe^{it})|^p dx \leq \frac{1}{2} \sec^p \left(\frac{\pi}{2p} \right) \int_0^{2\pi} |f(e^{i\theta})|^p d\theta \quad \text{for all} \quad t \in \mathbb{R}.$$

The inequality is sharp.

Furthermore, they conjectured that the inequality holds with the same sharp constant for $p > 2$ as well, which was later settled affirmatively by Melentijević and Božin [42]. Proceeding along this line, the present authors [14] considered the problem on a pair of diameters of $\mathbb{T}$, instead of the single diameter $-1 \leq x \leq 1$.

Theorem 12 ([14]). *If $f \in h^p$ for some $p > 1$, then the following inequality holds:*

$$\int_{D_0+D_1} |f(z)|^p |dz| \leq A_p(\theta) \int_{\mathbb{T}} |f(z)|^p |dz|, \tag{6}$$

where D_0, D_1 are two diameters of $\mathbb{T}$, θ is the smallest angle between these lines, and

$$A_p(\theta) = \begin{cases} \sec^p \left(\frac{\pi}{2p} \right) \frac{1}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}} & \text{if } 1 < p \leq 2, \\ \frac{2}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}} & \text{if } p \geq 2. \end{cases}$$

Surprisingly, Theorem 12 leads to the following inequalities for sequences, the latter of which contains the famous Hilbert's inequality as a special case.

Theorem 13 ([14]). *Suppose $\{a_n\}$ and $\{b_n\}$ are square summable sequences of real numbers, and θ is any acute angle. Let $k(\theta) = \frac{\pi}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}}$. Then, with $k + l$ restricted to be even,*

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(a_k a_l + b_k b_l)\cos(k - l)\frac{\theta}{2} + 2a_k b_l \cos(k + l)\frac{\theta}{2}}{k + l + 1} \leq k(\theta) \sum_{n=0}^{\infty} (a_n^2 + b_n^2), \tag{7}$$

and without this restriction,

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(a_k a_l + b_k b_l)\cos(k - l)\frac{\theta}{2} + 2a_k b_l \cos(k + l)\frac{\theta}{2}}{k + l + 1} \leq 2k(\theta) \sum_{n=0}^{\infty} (a_n^2 + b_n^2). \tag{8}$$

As mentioned, for $\theta = 0$ and $a_n = b_n$, inequality (8) reduces to Hilbert's inequality

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{a_k a_l}{k + l + 1} \leq \pi \sum_{n=0}^{\infty} a_n^2.$$

5.5 Open problems

We believe that inequality (6) is sharp for $1 < p \leq 2$. In fact, we expect (6) to hold with the sharp constant

$$A_p(\theta) = \sec^p \left(\frac{\pi}{2p} \right) \frac{1}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}}$$

for every $p > 1$. We pose this as a conjecture.

Conjecture 1. *Let $f \in h^p$ for some $p > 1$. Then the following sharp inequality holds:*

$$\int_{D_0+D_1} |f(z)|^p |dz| \leq \sec^p\left(\frac{\pi}{2p}\right) \frac{1}{\sin\frac{\theta}{2} + \cos\frac{\theta}{2}} \int_{\mathbb{T}} |f(z)|^p |dz|, \qquad (9)$$

where D_0, D_1 are two diameters of $\mathbb{T}$ and θ is the acute angle between them.

Comparison of the integrals of $|f|^p$ along different curves arises naturally in the context of Hardy spaces. Inequalities of the form

$$\int_C |f(z)|^p |dz| \leq A \int_\Gamma |f(z)|^p |dz|,$$

for f analytic and C lying inside Γ, hold with sharp constants

(a) $A = 1$ if Γ and C are circles;
(b) $A = 2$ if Γ is a circle and C is any convex curve;
(c) $A = (e+1)\pi + e$ if C and Γ are any convex curves.

These are connected with inequalities between bilinear or Hermitian forms. It may be of interest to obtain sharp analogues of these inequalities for harmonic functions, as one can expect them to have further implications like Theorem 13.

The harmonic Koebe function K plays the extremal role in many problems concerning univalent harmonic mappings. Therefore, in view of Theorem 9, one naturally asks whether the Baernstein type inequalities indeed hold for the whole class S_H^0.

Question 1. *Let $0 < p < \infty$. Do the inequalities*

$$M_p(r, h') \leq M_p(r, H_2') \quad \textit{and} \quad M_p(r, g') \leq M_p(r, G_2')$$

hold for every function $f = h + \bar{g} \in S_H^0$?

An affirmative answer to this would imply that $|a_n|$ and $|b_n|$ are $O(n^2)$, i.e., the harmonic analogue of the Bieberbach conjecture has the correct order of magnitude. This will be a significant development on the coefficient problem of Clunie and Sheil-Small.

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International Conference on Special Functions, Analytic Functions, Metrics, and Quasiconformality (ICSAMY 2024) (In honor of Professor S. Ponnusamy)

December 17–20, 2024.

Organized by: Department of Mathematics, Indian Institute of Technology Indore

SECOND ANNOUNCEMENT

Topics Include: Analysis and geometry of metric spaces, Quasiconformal and quasiregular mappings, Several complex variables, Special functions, Univalent harmonic functions.

Important Dates:
Early Registration: August 31, 2024 Abstract Submission: August 31, 2024
Late Registration: September 30, 2024

International Speakers Include: Yusuf Abu-Muhanna (Sharjah), Matthew Badger (USA), Árpád Baricz (Romania), Teodor Bulboaca (Romania), Michael Dorff (USA), Manzi Huang (China), Pekka Koskela (Finland), Yaxiang Li (China), Gaven Martin (New Zealand), Miodrag Mateljević (Serbia), Sudeb Mitra (USA), Antti Rasila (China-Finland), Frode Rønning (Norway), Nageswari Shanmugalingam (USA), Alexander Solynin (USA), Toshiyuki Sugawa (Japan), Vesna Todorčević (Serbia), Vyron Vellis (USA), Xiantao Wang (China), Brock Williams (USA), Hiroshi Yanagihara (Japan). For detailed information please visit the conference website: https://icsamy.iiti.ac.in

Contact: Swadesh Kumar Sahoo (Convener), Sanjeev Singh (Co-convener), IIT Indore
Email: icsamy@iiti.ac.in

The 11th International Conference on Mathematics and Computing (ICMC 2025)

January 9–11, 2025.

Organized by: Department of Mathematics, Indian Institute of Technology Bhilai

FIRST ANNOUNCEMENT

Topics of the conference include: Computational applied mathematics, including, but not limited to the broad topics of Operations Research, Numerical Analysis, Computational Fluid Mechanics, Soft Computing, Cryptology & Security Analysis, Image Processing, Big Data, Cloud Computing, Data Analytics, IoT, Pervasive Computing and other emerging areas of research. In addition to these, there will be several tracks which may be obtained from the conference homepage:

https://events.iitbhilai.ac.in/icmc2025/

Organizers and Contact Details: Arnab Patra, (arnab@iitbhilai.ac.in), Lakshmi Kanta Patra (lkpatra@iitbhilai.ac.in), Pawan Kumar Mishra (pawan@iitbhilai.ac.in)

There will be a workshop at the above venue:

A Three-day Workshop on Harmonic Mappings, Partial Differential Equations, and Applications (HMPDEA 2025)

January 7–9, 2025

Topics of the workshop include: Harmonic mappings in connection with univalent functions, Hardy spaces, special functions, and composition operators; Analysis of PDE, distribution theory, and Sobolev space.

Organizers and Contact Details: Pawan Kumar Mishra (pawan@iitbhilai.ac.in) and Swadesh Kumar Sahoo (swadesh.sahoo@iiti.ac.in)

Details of Workshop/Conferences in India

For details regarding Mathematics Training and Talent Search Programme
Visit: https://mtts.org.in/programme/mtts2021/

For details regarding Annual Foundation Schools, Advanced Instructional Schools, NCM Workshops, Instructional Schools for Teachers, Teacher's Enrichment Workshops
Visit: https://www.atmschools.org/

Name: 7th International Conference On Frontiers In Industrial And Applied Mathematics (FIAM-2024)
Date: November 13, 2024–November 14, 2024
Venue: Central University of Punjab, Bathinda, Punjab, India.
Visit: fiam2024.cup.edu.in

Name: International Conference On Matrix Analysis And Mathematical Modelling (MAMM 2024)
Date: November 29, 2024–December 1, 2024
Venue: Dr B R Ambedkar National Institute of Technology Jalandhar, Punjab, India.
Visit: www.nitj.ac.in/MAMM2024/

Name: International Conference on Matrix Special functions, Analytic functions, Metrics and Quasiconformality (ICSAMY2024)
Date: December 17, 2024–December 20, 2024
Venue: Indian Institute of Technology Indore, India.
Visit: https://icsamy.iiti.ac.in/

Name: International Conference on Recent Developments in Pure and Applied Mathematics (IRDPAM-2025)
Date: January 06, 2025–January 07, 2025
Venue: UCE-BIT Campus, Anna University, Tiruchirappalli - 24, Tamilnadu, India.
Visit: www.irdpam.com

Details of Workshop/Conferences Abroad

Name: 20 Years Of Anosov Representations
Date: October 7, 2024–October 11, 2024
Venue: Max Planck Institute For Mathematics In The Sciences, Leipzig, Germany.
Visit: www.mis.mpg.de/events/series/20-years-of-anosov-representations

Name: Quantum Groups And Representation Theory
Date: October 12, 2024–October 15, 2024
Venue: North Carolina State University, Raleigh, North Carolina, U.S.A.
Visit: sites.google.com/ncsu.edu/conf-quantum-groups-rep2024/home

Name: AIM Workshop: Albertson Conjecture And Related Problems
Date: October 14, 2024–October 18, 2024
Venue: American Institute of Mathematics, Pasadena, California. U.S.A.
Visit: aimath.org/workshops/upcoming/albertson/

Name: Recent Progress On Geometric Analysis And Riemannian Geometry
Date: October 21, 2024–October 25, 2024
Venue: SLMath 17 Gauss Way, Berkeley, California, U.S.A.
Visit: www.slmath.org/workshops/1111#overview_workshop

Name: SIAM Conference On Mathematics Of Data Science (MDS24)
Date: October 21, 2024–October 25, 2024
Venue: Hilton Atlanta, Atlanta, Georgia, U.S.A.
Visit: www.siam.org/conferences/cm/conference/mds24

Name: The 7th Mediterranean International Conference of Pure & Applied Mathematics And Related Areas (MICOPAM 2024)
Date: October 26, 2024–October 29, 2024
Venue: Antalya, TURKEY.
Visit: micopam.com

Name: 2nd International Conference On Differential Geometry (ICDG-FEZ'2024)
Date: October 28, 2024–October 31, 2024
Venue: Sidi Mohamed Ben Abdellah University, Faculty Of Sciences Dhar El Mahraz, Fez, Morocco.
Visit: www.fsdm.usmba.ac.ma/ICDGFEZ2024/

Name: AIM Workshop: Higher Du Bois And Higher Rational Singularities
Date: October 28, 2024–November 1, 2024
Venue: American Institute of Mathematics, Pasadena, California, U.S.A.
Visit: aimath.org/workshops/upcoming/higherdubois/

Name: Representation Theory Days (In Honor Of George Lusztig)
Date: November 9, 2024–November 11, 2024
Venue: Massachusetts Institute of Technology, Massachusetts, U.S.A.
Visit: math.mit.edu/events/representation-theory-days/

Name: AIM Workshop: Nilpotent Counting Problems In Arithmetic Statistics
Date: November 11, 2024–November 15, 2024
Venue: American Institute Of Mathematics, Pasadena, California, U.S.A.
Visit: aimath.org/workshops/upcoming/nilpotentarithstat/

Name: Geometry And Analysis Of Special Structures On Manifolds
Date: November 18, 2024–November 22, 2024
Venue: SLMath 17 Gauss Way, Berkeley, California, U.S.A.
Visit: www.slmath.org/workshops/1113#overview_workshop

Name: Masamu Advanced Study Institute (MASI) And Workshops
Date: November 22, 2024–December 1, 2024
Venue: University of Namibia, Windhoek, Namibia.
Visit: masamu.auburn.edu/

Name: 46th Australasian Combinatorics Conference
Date: December 2, 2024–December 6, 2024
Venue: The University of Queenslande, Brisbane, Australia.
Visit: 46acc.github.io/

Name: Asian Technology Conference In Mathematics
Date: December 8, 2024–December 11, 2024
Venue: Universitas Negeri Yogyakarta, Yogyakarta, Indonesia.
Visit: www.atcm.mathandtech.org/

Name: Hot Topics: Life After The Telescope Conjecture
Date: December 9, 2024–December 13, 2024
Venue: SLMath, 17 Gauss Way, Berkeley, California, U.S.A.
Visit: www.slmath.org/workshops/1103#overview_workshop

Name: AIM Workshop: Low-Degree Polynomial Methods In Average-Case Complexity
Date: December 9, 2024–December 13, 2024
Venue: American Institute Of Mathematics, Pasadena, California, U.S.A.
Visit: aimath.org/workshops/upcoming/lowdegreecomplexity/

Name: Second International Conference On Mathematics And Its Applications (ICMA-2024)
Date: December 13, 2024–December 15, 2024
Venue: Tribhuvan University, Kathmandu, Bagmati Province, Nepal.
Visit: icma2024.nms.org.np/

Name: International Conference on Fractional Calculus and Applications
Date: December 26, 2024–December 30, 2024
Venue: Sousse–Tunisia.
Visit: icofca.com/

Name: ACM-SIAM Symposium on Discrete Algorithms (SODA25) Held Jointly With SIAM Symposium On Algorithm Engineering And Experiments (ALENEX) SIAM Symposium On Simplicity In Algorithms (SOSA)
Date: January 12, 2025–January 15, 2025
Venue: Astor Crowne Plaza, New Orleans French Quarter, New Orleans, Louisiana, U.S.A.
Visit: www.siam.org/conferences/cm/conference/soda25

Name: Women In Mathematical Computational Biology
Date: January 13, 2025–January 17, 2025
Venue: Institute for Computational And Experimental Research In Mathematics, Brown University, Providence, RI, U.S.A.
Visit: icerm.brown.edu/topical_workshops/tw-25-wmcb/

Name: Patterns, Dynamics, and Data In Complex Systems
Date: January 21, 2025–January 24, 2025
Venue: Institute for Computational And Experimental Research In Mathematics, Brown University, Providence, RI, U.S.A.
Visit: icerm.brown.edu/topical_workshops/tw-25-pddcs/

Name: Connections Workshop: Probability And Statistics Of Discrete Structures
Date: January 23, 2025–January 24, 2025
Venue: SLMath, 17 Gauss Way, Berkeley, California, U.S.A.
Visit: www.slmath.org/workshops/1085#overview_workshop

Name: AIM Workshop: Motives And Mapping Class Groups
Date: January 27, 2025–January 31, 2025
Venue: American Institute Of Mathematics, Pasadena, California, U.S.A.
Visit: aimath.org/workshops/upcoming/motivesandmcg/

Name: Introductory Workshop: Probability And Statistics Of Discrete Structures
Date: January 27, 2025–January 31, 2025
Venue: SLMath 17 Gauss Way, Berkeley, California, U.S.A.
Visit: www.slmath.org/workshops/1086#overview_workshop

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